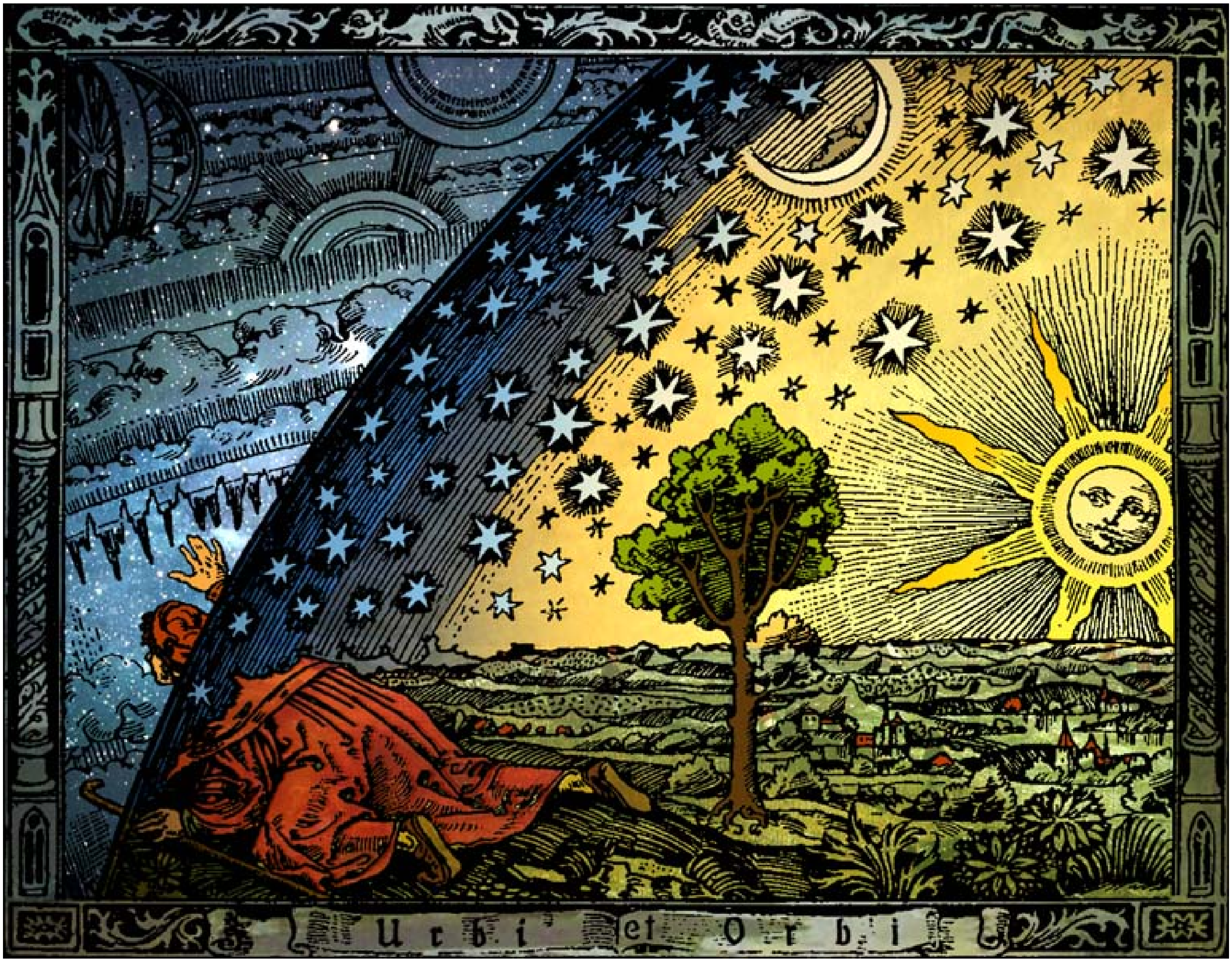

College Algebra

On the cover:

A colored version of the Flammarion Engraving. The black and white version of this engraving was included in the 1888 book by Camille Flammarion *L'Atmosphère - Météorologie Populaire*, which was a book for general audiences on meteorology. The original engraving was captioned: "Un missionaire du moyen âge raconte qu'il avait trouvé le point où le ciel et la Terre se touche..." This translates as "A missionary of the Middle Ages told of how he found the point where the sky and the earth touch..."

The Flammarion Engraving is often interpreted as representing the human quest to discover the inner workings of the universe. This colored version is from the Wikimedia Commons and is credited to Hugo Heikenwaelder.

College Algebra

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Contents

Chapter 1

Algebra Review

This College Algebra text will cover a combination of classical algebra and analytic geometry, with an introduction to the transcendental exponential and logarithmic functions. If mathematics is the language of science, then algebra is the grammar of that language. Like grammar, algebra provides a structure to mathematical notation, in addition to its uses in problem solving and its ability to change the appearance of an expression without changing the value.

1.1 Algebraic Simplification

When algebraic techniques are presented as skills in isolation, they are much simpler to understand and practice. However the problem solving process in any context involves deciding which skills to use when. Most College Algebra students will have practiced problems in the form:

$$(x+7)(x-2) = ?$$

or

$$(2x+1)^2 = ?$$

The problems in this section deal with a combination of these processes which are often encountered as parts of more complex problems.

Examples

Simplify:

$3(x-1)(2x+5)-(x+4)^2$

In this example, the simplification involves two expressions: $3(x-1)(2x+5)$ and $(x+4)^2$. The $(x+4)^2$ is preceded by a negative (or subtraction) sign. This textbook will often treat $-x$ and $+(-x)$ as equivalent statements, since subtraction is defined as the addition of a negative.

We will simplify each expression separately and then look to combine like terms.

$$3(x-1)(2x+5)-(x+4)^2 = 3(2x^2+3x-5)-(x^2+8x+16)$$

Notice that the results of both multiplications remain inside of parentheses. This is because each one has something that must be distributed.

In the case of $(2x^2+3x-5)$, there is a 3 which must be distributed, resulting in $6x^2+9x-15$. In the case of $(x^2+8x+16)$ there is a negative sign or -1 which must be distributed, resulting in $-x^2-8x-16$. It is important in these situations that the negative sign be distributed to all terms in the parentheses.

So:

$$\begin{aligned} 3(x-1)(2x+5)-(x+4)^2 &= 3(2x^2+3x-5)-(x^2+8x+16) \\ &= 6x^2+9x-15-x^2-8x-16 \\ &= 5x^2+x-31 \end{aligned}$$

Simplify:

$2(x+3)^2-4(3x-1)(x+2)$

This example shows some of the same processes as the previous example. There are again two expressions that must be simplified, each of which has a coefficient that must be distributed. It is often helpful to wait until after multiplying the binomials before distributing the coefficient. However, as is often true in mathematics, there are several different approaches that may be taken in simplifying this problem.

If someone prefers to first distribute the coefficient before multiplying the binomials, then the coefficient must only be distributed to ONE of the binomials, but not both. For example, in multiplying $3*2*5 = 30$, we can first multiply $2*5 = 10$ and then $3 * 10 = 30$. Each factor is multiplied only once.

In the example above we can proceed as we did with the previous example:

$$\begin{aligned} 2(x+3)^2 - 4(3x-1)(x+2) &= 2(x^2+6x+9) - 4(3x^2+5x-2) \\ &= 2x^2+12x+18-12x^2-20x+8 \\ &= -10x^2-8x+26 \end{aligned}$$

Or, we can choose to distribute the 4 first:

$$\begin{aligned} 2(x+3)^2 - 4(3x-1)(x+2) &= 2(x^2+6x+9) - (12x-4)(x+2) \\ &= 2x^2+12x+18-(12x^2+20x-8) \\ &= 2x^2+12x+18-12x^2-20x+8 \\ &= -10x^2-8x+26 \end{aligned}$$

Or, we can distribute the 4 as a negative. If we do this, then the sign in front of the parentheses will be positive:

$$\begin{aligned} 2(x+3)^2 - 4(3x-1)(x+2) &= 2(x^2+6x+9) + (-12x+4)(x+2) \\ &= 2x^2+12x+18+(-12x^2-20x+8) \\ &= -10x^2-8x+26 \end{aligned}$$

Distributing the 2 in front of the squared binomial must also be handled carefully if you choose to do this. If you distribute the 2 before squaring the $(x+3)$, then the 2 will be squared as well. If you choose to distribute the 2, the $(x+3)^2$ must be written out as $(x+3)(x+3)$:

$$\begin{aligned}
2(x+3)^2 - 4(3x-1)(x+2) &= 2(x+3)(x+3) - 4(3x-1)(x+2) \\
&= (2x+6)(x+3) - 4(3x^2+5x-2) \\
&= 2x^2 + 12x + 18 - 12x^2 - 20x + 8 \\
&= -10x^2 - 8x + 26
\end{aligned}$$

Most examples in this text will distribute the coefficients as the last step before combining like terms for a final answer.

Simplify:

$3x[5-(2x+7)] + (3x-2)^2 - (x-5)(x+4)$

This example has three expressions that should be simplified separately before combining like terms. In the first expression $3x[5-(2x+7)]$, we should simplify inside the brackets before distributing the $3x$.

$$3x[5-(2x+7)] + (3x-2)^2 - (x-5)(x+4)$$

$$\begin{aligned}
&= 3x[5-2x-7] + (3x-2)^2 - (x-5)(x+4) \\
&= 3x[-2x-2] + (3x-2)(3x-2) - (x^2-x-20) \\
&= -6x^2 - 6x + (9x^2 - 12x + 4) - x^2 + x + 20 \\
&= -6x^2 - 6x + 9x^2 - 12x + 4 - x^2 + x + 20 \\
&= 2x^2 - 17x + 24
\end{aligned}$$

Exercises 1.1

Simplify each expression.

1) $(x-2)[2x-2(3+x)]-(x+5)^2$

2) $3x^2-[7x-2(2x-1)(3-x)]$

3) $(a+b)^2-(a+b)(a-b)-[a(2b-2)-(b^2-2a)]$

4) $5x-3(x-2)(x+7)+3(x-2)^2$

5) $(m+3)(m-1)-(m-2)^2+4$

6) $(a-1)(a-2)-(a-2)(a-3)+(a-3)(a-4)$

7) $2a^2-3(a+1)(a-2)-[7-(a-1)]^2$

8) $2(x-5)(3x+1)-(2x-1)^2$

9) $6y+(3y+1)(y+2)-(y-3)(y-8)$

10) $6x-4(x+10)(x-1)+(x+1)^2$

1.2 Factoring

This section will review three of the most common types of factoring - factoring out a Greatest Common Factor, Trinomial Factoring and factoring a Difference of Squares.

Greatest Common Factor

Factoring out a greatest common factor essentially undoes the distributive multiplication that often occurs in mathematical expressions. This factor may be monomial or polynomial, but in these examples, we will explore monomial common factors.

In multiplying $3xy^2(5x-2y) = 15x^2y^2-6xy^3$ the monomial term $3xy^2$ is multiplied or distributed to both terms inside the parentheses. The process of factorization undoes this multiplication.

Example:

Factor $7x^2 + 14x$

This expression has two terms. The coefficients share a common factor of 7 and the only variable involved in this expression is x. The highest power of the variable that is shared by both terms is x^1, so this is the power of x that can be factored out of both terms. The greatest common factor is $7x$.

$$7x^2 + 14x = 7x(x + 2)$$

It isn't necessary to find the greatest common factor right away. In more complicated problems, the factoring can be accomplished in pieces, similar in fashion to reducing fractions.

Example:

Factor $42x^2y^6 + 98xy^3 - 210x^3y^2$

This expression has three terms. It's not immediately clear what the greatest common factor of the coefficients is, but they're all even numbers, so we could at least divide them all by 2. The $98xy^3$ term has an x^1, which means that this is the hightest power of x that we could factor out of all the terms. The $210x^3y^2$ has a y^2, which is the highest power of y that can be factored out of all the terms. So we can at least proceed with these factors:

$$\begin{aligned} 42x^2y^6 + 98xy^3 - 210x^3y^2 &= 2xy^2 * 21xy^4 + 2xy^2 * 49y - 2xy^2 * 105x^2 \\ &= 2xy^2(21xy^4 + 49y - 105x^2) \end{aligned}$$

Now, we didn't try very hard to find the greatest common factor in the beginning of this problem, so it's important that we continue to question whether or not there are any remaining common factors. The 21 and 49 clearly share a common factor of 7, so it would make sense to see if 105 is divisible by 7 as well. If we divide 105 by 7, we see that 105 = 7*15. So, we can also factor out a common factor of 7 from the remaining terms in the parentheses.

$$\begin{aligned} 2xy^2(21xy^4 + 49y - 105x^2) &= 2xy^2(7 * 3xy^4 + 7 * 7y - 7 * 15x^2) \\ &= 7 * 2xy^2(3xy^4 + 7y - 15x^2) \\ &= 14xy^2(3xy^4 + 7y - 15x^2) \end{aligned}$$

Trinomial Factoring ($a = 1$)

Trinomial factoring undoes the multiplication of two binomials, and it comes in two flavors - simple and complex. The simplest form of trinomial factoring involves a trinomial expression in the form $ax^2 + bx + c$ in which the value of a is 1. This makes the task of factorization simpler than if the value of a is not 1.

Example

Factor $x^2 + 7x + 10$

In this example, the value of a is 1, which makes this type of trinomial factoring a little less difficult that it would otherwise be. Whether or not the value of a is 1, the fundamental issue that governs this type of factoring is the $+$ or $-$ sign of the constant term. In this problem, the constant term is positive. That means that we need to find factors of 10 that add up to 7. This is relatively straightforward:

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

A companion problem to this one is $x^2 - 7x + 10$. Notice that, in this case, the sign of the constant term is still positive, which means that we still need factors of 10 that add up to 7. This means we still need to use 2 and 5. However, in this case, instead of the +10 being produced from a multiplication of $(+2)(+5)$, it is the result of multiplying $(-2)(-5)$. This is what makes the 7 in the second example negative:

$$x^2 - 7x + 10 = (x - 2)(x - 5)$$

Example

Factor $x^2 + 3x - 10$

In this case, the sign of the constant term is negative. That means that we need to find factors of 10 that have a difference of 3. This is still 5 and 2.

$$x^2 + 3x - 10 = (x - 2)(x + 5)$$

The multiplication of the (-2) and the $(+5)$ produce the (-10) and the fact that the 2 and 5 have opposite signs creates the difference that gives us $(+3)$. A companion problem to this one is $x^2 - 3x - 10$. In this case, the sign of the constant term is still negative, which means that we still need factors of 10 that have a difference of 3. This means we still need to use 2 and 5. However in this case, instead of the $(+3)$ as the coefficient of the middle term, we'll need a (-3). To do this we simply reverse the signs of the 2 and 5 from the previous problem:

$$x^2 - 3x - 10 = (x + 2)(x - 5)$$

Now the $(+2)(-5)$ gives us (-10), but the $+2x - 5x$ gives us $(-3x)$ instead of $(+3x)$.

Example

Factor $x^2 + 11x - 42$

In this problem, the sign of the constant term is negative. That means that we need factors of 42 that have a difference of 11. A systematic exploration of all the factor pairs of 42 can help us to find the correct pair:

1	42
2	21
3	14
6	7

Here, we can see that the factors 3 and 14 have a difference of 11. This means that we will use these factors in our answer: $(x \quad 3)(x \quad 14)$. In determining how to place the $+$ and $-$ signs in the parentheses, we can refer back to the original problem: $x^2+11x-42$. If we want a difference of $(+11x)$, then we'll need to have a $(+14)$ and a (-3):

$$x^2 + 11x - 42 = (x - 3)(x + 14)$$

Example

Factor $x^2 + 28x + 96$

In this problem, the sign of the constant term is positive. That means that we need factors of 96 that add up to 28. A systematic exploration of all the factor pairs of 96 can help us to find the correct pair:

1	96
2	48
3	32
4	24
6	16
8	12

Here, we can see that the factors 4 and 24 add up to 28. This means that we will use these factors in our answer: $(x \quad 4)(x \quad 24)$. In determining how to place the $+$ and $-$ signs in the parentheses, we can refer back to the original problem: $x^2 + 28x + 96$. If we want 4 and 24 to add up to $(+28)$, then they should both be positive:

$$x^2 + 28x + 96 = (x + 4)(x + 24)$$

In building the charts of factor pairs in the previous two problems, nothing more difficult than dividing the constant term by the numbers 1, 2, 3, 4, 5, 6, ... and so on can help you to find the full list of factor pairs. If you don't get a whole number when dividing - for instance $96 \div 5 = 19.2$, then this number is not included in the list of factor pairs.

Trinomial Factoring ($a \neq 1$)

If the value of a is not 1, this means that, if the trinomial is factorable, at least one of its binomial factors also has a coefficient other than 1. For instance:

$$(2x + 7)(x - 3) = 2x^2 + 1x - 21$$

If we were to try undo this multiplication through the process of trinomial factoring, we should look to the sign of the constant term. In this example, the sign is negative. This still means that we will need to find factor pairs that produce a difference of $(+1x)$ as the middle term. However, in this scenario, it is not just the factors of 21 that are involved in producing the $(+1x)$, but the combination of the factors of 21 and the factors of the leading coefficient 2. The middle term $(+1x)$ comes from the multiplication of the $(2x)(-3)$ and the multiplication of $(+7)(+1x)$:

$$(2x + 7)(x - 3)$$

$$\begin{aligned}(2x + 7)(x - 3) &= 2x^2 - 6x + 7x - 21 \\ &= 2x^2 + x - 21\end{aligned}$$

In trying to factor a trinomial like $2x^2 + x - 21$, we need to take this into consideration. For example, if we were to factor $3x^2 - 10x + 8$, we should first still look to the sign of the constant term, which, in this case, is positive. That means we want factor pairs that will add up to 10. But we have to take into consideration the interaction of the factors of the 3 with the factors of the 8. The 3 is a prime number, which means that we don't have a choice - it can only be split up into $3 * 1$, so we can start:

Factor $3x^2 + 10x + 8$

$$(3x \quad ?)(x \quad ?)$$

Our options for filling in the question marks will come from the factors of 8, either $8 * 1$ or $4 * 2$. The process is by trial and error:

$$(3x \quad 8)(x \quad 1)$$

$$3x + 8x = 11x$$

$$(3x \quad 1)(x \quad 8)$$

$$24x + 1x = 25x$$

$$(3x \quad 4)(x \quad 2)$$

$$6x + 4x = 10x$$

$$(3x \quad 2)(x \quad 4)$$

$$12x + 2x = 14x$$

We can see that the choice above:

$$(3x \quad 4)(x \quad 2)$$

gives us the required $10x$ as the middle term. Since the original problem was $3x^2 + 10x + 8$ we'll want to fill in the signs as both positive:

$$3x^2 + 10x + 8 = (3x + 4)(x + 2)$$

A second method for handling this type of factoring depends on how the factors of the leading coefficient and constant term interact with each other to produce the middle term. In this process, given the problem $3x^2 + 10 + 8$, we can multiply the first and last coefficient and then look at the factor pairs of the product:

$$3 * 8 = 24$$

1	24
2	12
3	8
4	6

We can see that the factor pair of 24 that adds up to 10 is $6 * 4$. We proceed by splitting the $10x$ into $6x + 4x$ and then factor by grouping. If you are uncomfortable with factoring by grouping, then this is probably not a good method to try. However, if you are comfortable with factoring by grouping, the rest of the process is relatively straightforward:

$$3x^2 + 10x + 8 = 3x^2 + 6x + 4x + 8$$

We then factor a common factor from the first two terms and the last two terms separately, and then factor out the common binomial factor of $(x + 2)$:

$$\begin{aligned} 3x^2 + 10x + 8 &= 3x^2 + 6x + 4x + 8 \\ &= 3x(x + 2) + 4(x + 2) \\ &= (x + 2)(3x + 4) \end{aligned}$$

Example

Factor $7x^2 - 5x - 18$

In this example, the sign of the constant term is negative, which means that we'll need factor pairs that produce a difference of 5. The leading coefficient is 7, which is prime, so, again, the only way to split up the 7 is $7 * 1$.

$$(7x \quad ?)(x \quad ?)$$

The options for filling in the question marks come from the factors of 18, for which there are three possibilities: $18 * 1$, $9 * 2$, or $6 * 3$. We'll try each of these factor pairs in place of the question marks:

$$(7x \quad 18)(x \quad 1)$$
$$18x - 7x = 11x$$

$$(7x \quad 1)(x \quad 18)$$
$$126x - 1x = 125x$$

$$(7x \quad 9)(x \quad 2)$$
$$14x - 9x = 5x$$

$$(7x \quad 2)(x \quad 9)$$
$$63x - 2x = 61x$$

$$(7x \quad 6)(x \quad 3)$$
$$21x - 6x = 15x$$

$$(7x \quad 3)(x \quad 6)$$
$$42x - 3x = 39x$$

The choice above:

$$(7x \quad 9)(x \quad 2)$$

gives us the required $5x$ as the middle term. Since we're looking for a $(-5x)$, we'll make the 14 negative and the 9 positive: $-14x + 9x = -5x$.

$$7x^2 - 5x - 18 = (7x + 9)(x - 2)$$

If we want to try the other method for factoring $7x^2 - 5x - 18$, we would multiply $7 * 18 = 126$, and then work to find factor pairs of 126 that have a difference of 5:

1	126
2	63
3	42
6	21
7	18
9	14

Here, the last factor pair, $9 * 14$, has a difference of 5. So then we proceed to factor by grouping:

$$\begin{aligned} 7x^2 - 5x - 18 &= 7x^2 + 9x - 14x - 18 \\ &= x(7x + 9) - 2(7x + 9) \\ &= (7x + 9)(x - 2) \end{aligned}$$

Notice that when the -2 was factored out from the last two terms $-14x - 18$, we ended up with $-2(7x + 9)$, because $(-2) * (+9) = -18$. This is also important because in order to factor out the common binomial factor of $(7x + 9)$, this binomial must be exactly the same in both terms.

Difference of Squares

Factoring a difference of squares is actually a special form of trinomial factoring. If we consider a trinomial of the form $ax^2 + bx + c$, where c is a perfect square and negative, we will find something interesting about the possible values of b that make the trinomial factorable.

Example

Consider $x^2 + bx - 36$

For this expression to be factorable, the middle coefficient b would need to be equal to the difference of any of the factor pairs of 36. If we look at the possible factor pairs, we see the following:

1	36
2	18
3	12
4	9
6	6

This means that the possible values for b that would make this expression factorable are:

$$36 - 1 = 35 \to x^2 + 35x - 36 = (x + 36)(x - 1)$$

$$18 - 2 = 16 \to x^2 + 16x - 36 = (x + 18)(x - 2)$$

$$12 - 3 = 9 \to x^2 + 9x - 36 = (x + 12)(x - 3)$$

$$9 - 4 = 5 \to x^2 + 5x - 36 = (x + 9)(x - 4)$$

$$6 - 6 = 0 \to x^2 + 0x - 36 = x^2 - 36 = (x + 6)(x - 6)$$

As we see, factoring $x^2 - 36$ means that the factors of the perfect square $36 = 6 * 6$ will cancel each other out leaving $0x$ in the middle. If there is a perfect square as the leading coefficient, then this number should be square rooted as well:

$$16x^2 - 25 = (4x + 5)(4x - 5)$$

In the example above, the $+20x$ and $-20x$ as the middle terms cancel each other out leaving just $16x^2 - 25$.

These three types of factoring can also be combined with each other as we see in the following examples.

Example

Factor $2x^2 - 50$

This is not a trinomial because it doesn't have three terms. It is also not a difference of squares because 2 and 50 are not perfect squares. However, there is a common factor of 2 which we can factor out:

$$2x^2 - 50 = 2(x^2 - 25)$$

The expression inside the parentheses is a difference of squares and should be factored:

$$2x^2 - 50 = 2(x^2 - 25) = 2(x + 5)(x - 5)$$

Example

Factor $24 - 2x - x^2$

Here the sign of the x^2 term is negative. For this problem we can factor out a -1 and proceed as we did with the previous problems in which the leading coefficient was positive or we can factor it as it is:

$$24 - 2x - x^2 = -(x^2 + 2x - 24) = -(x + 6)(x - 4)$$

If we want to factor it as it is, we should be aware that the constant term is positive and the quadratic term is negative, which means that we will want the factors of 24 to have a difference of 2.

$$24 - 2x - x^2 = (6 + x)(4 - x)$$

Example

Factor $6x^2 + 12x + 6$

First, we notice that this expression has a common factor of 6. If we factor out the 6, then we should be left with an easier problem:

$$6x^2 + 12x + 6 = 6(x^2 + 2x + 1) = 6(x + 1)(x + 1) = 6(x + 1)^2$$

Exercises 1.2

Factor each expression completely.

1) $8a^2b^3 + 24a^2b^2$

2) $19x^2y - 38x^2y^3$

3) $13t^8 + 26t^4 - 39t^2$

4) $5y^5 + 25y^4 - 20y^3$

5) $45m^4n^5 + 36mn^6 + 81m^2n^3$

6) $125x^3y^5 + 60x^4y^4 - 85x^5y^2$

Factor each trinomial into the product of two binomials.

7) $a^2 + 3a + 2$

8) $y^2 - 8y - 48$

9) $x^2 - 6x - 27$

10) $t^2 - 13t + 42$

11) $m^2 + 3m - 54$

12) $x^2 + 11x + 24$

Factor completely. Remember to look first for a common factor. If the polynomial is prime, state this.

13) $a^2 - 9$

14) $y^2 - 121$

15) $-49 + k^2$

16) $-64 + t^2$

17) $6x^2 - 54$

18) $25y^2 - 4$

19) $200 - 2a^2$

20) $3m^2 - 12$

21) $98 - 8k^2$

22) $-80w^2 + 45$

23) $5y^2 - 80$

24) $-4a^2 + 64$

25) $8y^2 - 98$

26) $24a^2 - 54$

27) $36k - 49k^3$

28) $16y - 81y^3$

Factor each trinomial completely. Remember to look first for a common factor. If the polynomial is prime, state this.

29) $3y^2 - 15y + 16$

30) $8a^2 - 14a + 3$

31) $9x^2 - 18x + 8$

32) $6a^2 - 17a + 12$

33) $2x^2 + 7x + 6$

34) $2m^2 + 13m - 18$

35) $20y^2 + 22y + 6$

36) $36x^2 + 81x + 45$

37) $24a^2 - 42a + 9$

38) $48x^2 - 74x - 10$

Factor each expression completely.

39) $30 + 7y - y^2$

40) $45 + 4a - a^2$

41) $24 - 10x - x^2$

42) $36 - 9x - x^2$

43) $84 - 8x - x^2$

44) $72 - 6a - a^2$

45) $6y^2 + 24y + 15$

46) $10y^2 - 75y + 35$

47) $20ax^2 - 36ax - 8a$

1.3 Quadratic Equations

Quadratic equations are equations of the second degree. The solution of quadratic equations has a long history in mathematics going back several thousand years to the geometric solutions produced by the Babylonian culture. The Indian mathematician Brahmagupta used "rhetorical algebra" (algebra written out in words) in the 7th century to produce solutions to quadratic equations and Arab mathematicians of 9th and 10th centuries followed simlar methods. Leonardo of Pisa, also known as Fibonacci included information on the Arab approach to solving quadratic equations in his book *Liber Abaci*, published in 1202.

The quadratic formula is generally used to solve quadratic equations in standard form: $ax^2 + bx + c = 0$. The solutions for this are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now, the question is - why does this formula give solutions to the standard quadratic equation? We can proceed as we normally do in solving linear equations - that is, by getting the x by itself. The only problem here is that instead of just x, there are also terms involving x^2. This is where the process of completing the square comes in handy.

We can begin with the quadratic equation in standard form:

$$ax^2 + bx + c = 0$$

Just as it is easier to factor a quadratic trinomial if the leading coefficient is 1, this process of completing the square is also easier if the leading coefficient is 1. So, next we will divide through on both sides of this equation by a.

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$

$$x^2 + \tfrac{b}{a}x + \tfrac{c}{a} = 0$$

Then, we will move the $\frac{c}{a}$ to the other side of the equation to clear out some room for completing the square:

$$\begin{aligned} x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ -\frac{c}{a} &= -\frac{c}{a} \\ x^2 + \frac{b}{a}x \qquad &= -\frac{c}{a} \end{aligned}$$

Now we need to complete the square. If you are already familiar with this process, you may wish to skip the following explanation.

If we look at what happens when we square a binomial like $(x+3)^2$, we will begin to notice a pattern.

$$(x+3)^2 = (x+3)(x+3) = x^2 + 6x + 9$$

$$(x+4)^2 = (x+4)(x+4) = x^2 + 8x + 16$$

$$(x+5)^2 = (x+5)(x+5) = x^2 + 10x + 25$$

$$(x+6)^2 = (x+6)(x+6) = x^2 + 12x + 36$$

Our goal in the derivation of the Quadratic Formula is to rewrite the expression $x^2 + \frac{b}{a}x$ as a perfect square in the form $(x+ \quad)^2$. The reason that we want to do this is that writing an expression as a binomial squared eliminates the problem of having both an x and an x^2, which was preventing us from getting the x by itself in the standard quadratic equation.

If we can figure out what should take the place of the blanks in the statement:

$$x^2 + \tfrac{b}{a}x + \qquad = (x+ \qquad)^2$$

then we will be well on our way to deriving the quadratic formula.

If we re-examine the sample perfect binomial squares from the previous page, we note a useful pattern. This is that the blank in the parentheses $(x + \quad)^2$ is filled by a number that is one-half the value of the linear coefficient - or the coefficient of the x^1 term. Notice that in $x^2 + 6x + 9 = (x+3)^2$, 3 is half of 6, in $x^2 + 8x + 16 = (x+4)^2$, the 4 is half of 8, and so on. If we want to write $x^2 + \frac{b}{a}x + \quad$ as a perfect square in the form $(x + \quad)^2$, the blank in the parentheses should be filled by:

$$\frac{1}{2} * \frac{b}{a} = \frac{b}{2a}$$

Now, it's not true that $x^2 + \dfrac{b}{a}x + \quad = \left(x + \dfrac{b}{2a}\right)^2$

We're missing the constant term on the left. However, if we return to our perfect square examples, we can see that the constant term is always the square of the term inside inside the parentheses. So, we can restate our problem now as:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

So, if we return to our originial problem, we were saying that:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$-\frac{c}{a} = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x \qquad = -\frac{c}{a}$$

We can add $\frac{b^2}{4a^2}$ to both sides of this equation and then restate the left hand side as a perfect square of a binomial:

$$x^2 + \frac{b}{a}x \qquad = -\frac{c}{a}$$

$$+\frac{b^2}{4a^2} = +\frac{b^2}{4a^2}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

The last tricky bit of this derivation is adding the two fractions on the right hand side. The common denominator for these fractions is $4a^2$, so we'll need to multiply the $-\frac{c}{a}$ by $\frac{4a}{4a}$ to get $-\frac{4ac}{4a^2}$. Then the right hand side will be $\frac{b^2 - 4ac}{4a^2}$:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Then, we can take the square root of both sides and get the x by itself:

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Subtracting $\frac{b}{2a}$ from both sides is easy since we already have a common denominator:

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$-\frac{b}{2a} = -\frac{b}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

At the college algebra level, it is often useful to program the quadratic formula onto a graphing calculator both for easy use and also to learn a little bit about programming. The following program is a simple example of this for the TI-84 series of graphing calculators. Graphing calculators also often have built-in polynomial solver feature that can be used to solve quadratics.

Press the "prgm" key in the top middle of the calculator keypad. This will bring up a screen that shows EXEC EDIT NEW across the top. Arrow over across the top to "NEW," and then select 1: Create New.

This will bring up a screen asking you to name the program. You should see PROGRAM and then underneath it, "Name=". The alpha lock is on automatically, so any key you press will type the letter associated with it. Name your program and press ENTER. You should see PROGRAM: Name, with whatever name you've chosen for your program. Underneath this you will se a colon **:**. This is where you will enter the commands for the program.

First we need to enter the values for A, B and C from the quadratic equation into the calculator. To do this, press the "prgm" key again. Across the top of the screen you should see CTL I/O COLOR EXEC.

Arrow over to I/O. This is the "input/output" menu. Choose number 2:Prompt. This will return you to the program screen where you will see :Prompt under the name of the program. After :Prompt, type A, B, C. You'll need to use the "alpha" key to access the letters and the comma is right above the 7 key.

PROGRAM: Name (whatever name you've chosen should show here)
:Prompt A, B, C

On the next line of the program, we will take the values of A, B and C and use them to calculate the values of the roots of the equation. Type in the following:

PROGRAM: Name
:Prompt A, B, C
:$(-B + \sqrt{(B^2 - 4AC)})/(2A) \rightarrow R$
:$(-B - \sqrt{(B^2 - 4AC)})/(2A) \rightarrow S$

In typing these two lines it's important that when you type $-B$, you use the negative key next to the decimal point, rather than the subtraction key. The calculator is very picky about this. When you're typing the $B^2 - 4AC$, you'll need to use the subtraction key on the far right of the keypad.

Also notice the double parentheses - one set for the numerator of the fraction and one set for the square root. If you don't type this in correctly it will produce wrong answers. The arrow in the formula stores the values of the answer in the variables R and S, and the arrow is produced by the "sto→" key just above the ON button in the lower left of the keyboard.

Now that we've given the calculator the values for A, B and C and then had the calculator find the roots of the eqatiuon, we need to display the answers. If you press the prgm key and arrow over to the I/O menu again, you can choose 3:Disp. This will Display the answers that we've stored as R and S.

PROGRAM: Name
:Prompt A, B, C
:$(-B + \sqrt{B^2 - 4AC}))/(2A) \rightarrow R$
:$(-B - \sqrt{B^2 - 4AC}))/(2A) \rightarrow S$
:Disp R,S

Now we can test the program with some simple equations. To run the program, press the program key and choose the program you've created either by selecting it and pressing enter, or by pressing the number for the program in the list. This should bring you back to the calculation screen, where you can run the program by pressing enter. The calculator should then ask you for the values of A, B and C.

Solve for x: $2x^2 - x - 1 = 0$

In this example the values for A, B and C are:
$A = 2$
$B = -1$
$C = -1$

Again, it's important that you use the negative sign key next to the decimal point for the values of any negative coefficients and not the subtraction key. The calculator should return values of 1 and -0.5 as the solutions.

Solve for x: $x^2 + x + 1 = 0$

In this example the values for A, B and C are:
$A = 1$
$B = 1$
$C = 1$

The calculator should return values of $-0.5 \pm 0.8660254038i$ as solutions.

If you get an error message saying "NONREAL ANSWERS," you'll need to adjust the calculator setting to allow for complex valued answers. You can do this by presing the "mode" key in the top left of the keypad and arrowing down to the line that reads "REAL a+bi reˆ(θi)." You can then arrow over to "a+bi" and press enter. This will allow the calculator to compute complex valued answers.

Something very important to remember about the quadratic formula is that the equation must be in standard form in order to identify the values of A, B and C to use in the formula. For example in the equation:

$$3x^2 - 7 = 2x$$

it is important to understand that the values of A, B and C come from the standard form of the equation and not the present form of the equation. There are several pitfalls to watch out for in this equation. First of all, the $2x$ is on the opposite side of the equation from the other terms. That means that the value of B IS NOT +2. Also, if we were to move the $2x$ to the other side to put the equation in standard form, it is not the order of the terms, but degree of the variable that determines whether a coefficient is identified as A, B or C.

In moving the $2x$ to the other side of the equation, I have seen students put the term they're adding to that side as the last term. There is nothing wrong about this, but if you do that, you must be careful about identifying the values of A, B and C.

$$3x^2 - 7 = 2x$$

$$-2x = -2x$$

$$3x^2 - 7 - 2x = 0$$

There is nothing wrong about the way the equation above is written depsite the fact that it is not in "standard form." The important thing to remember is that "A" is ***not*** the coefficient of whichever term is listed first. It is the coefficient of the quradratic, or x^2 term. Likewise, "B" is not the coefficient of the second term, but rather the coefficient of the linear, or x^1 term. And "C" is not whichever number comes last, but rather the value of the constant term. So in the equation above, however it is written, the value of A is $+3$, B is -2 and C is -7.

Exercises 1.3

Solve for x in each equation. Round any irrational values to the nearest 1000th.

1) $x^2 + 7x = 2$

2) $5x^2 - 3x = 4$

3) $\frac{3}{4}x^2 = \frac{7}{8}x + \frac{1}{2}$

4) $\frac{2}{3}x^2 - \frac{1}{3} = \frac{5}{9}$

5) $2x^2 + (\sqrt{5})x - 3 = 0$

6) $3x^2 + x - \sqrt{2} = 0$

7) $2.58x^2 - 3.75x - 2.83 = 0$

8) $3.73x^2 + 9.74x + 2.34 = 0$

9) $5.3x^2 + 7.08x + 1.02 - 0$

10) $3.04x^2 + 1.35x + 1.234 = 0$

11) $7x(x + 2) + 5 = 3x(x + 1)$

12) $5x(x - 1) - 7 = 4x(x - 2)$

13) $14(x - 4) - (x + 2) = (x + 2)(x - 4)$

14) $11(x - 2) + (x - 5) = (x + 2)(x - 6)$

1.4 Complex Numbers

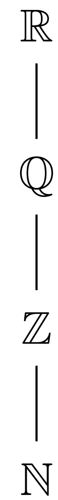
Our number system can be subdivided in many different ways. The most basic form of mathematics is counting and almost all human cultures have words to represent numbers (the Pirahã of South America are a notable exception). Thus the most basic set of numbers is the set of counting numbers represented by the double barred N: $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \dots\}$ (we will set aside the debate as to whether or not zero should be included in this set).

If we try to subtract a larger counting number from a smaller counting number we find that there are no members in the set of counting numbers to represent the answer in this situation. This extends the set of natural numbers to the set of integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. The integers are represneted by the double barred Z, for the German word for numbers - "zahlen." In the earliest appearances of negative numbers in the Chinese and Indian mathematical systems, negative values were often used to represent debt. Because Greek mathematics was based on Geometry, they did not use negative numbers.

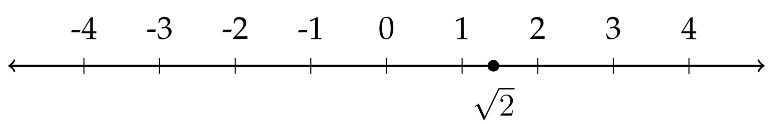
Moving to multiplication and division, if we question the value of $8 \div 2 = 4$ versus $8 \div 3 = ?$, we once again must expand our conception of numbers to allow for an answer to the second question $8 \div 3 = ?$. Understanding ratios of whole numbers or Rational numbers allows solutions to such problems. The set of Rational numbers is represented by the double barred Q, to represent a quotient: $\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \right\}$.

The Greek understanding of numbers mostly stopped here. They felt that all quantities could be represented as the ratio of whole numbers. The length of the diagonal of a square whose sides are of length 1 produced considerable consternation among the Pythagoreans as a result of this. Using the Pythagorean Theorem for the diagonal of a square whose sides are of length 1 shows that the diagonal would be $c^2 = 1^2 + 1^2 = 2$, thus $c = \sqrt{2}$. This number cannot be represented as a ratio of whole numbers. This new class of numbers adds the set of irrational numbers to the existing set of rational numbers to create the Real numbers, represented with a double barred R: $\mathbb{R}$.

This hierarchy of numbers is often represented in the following diagram:



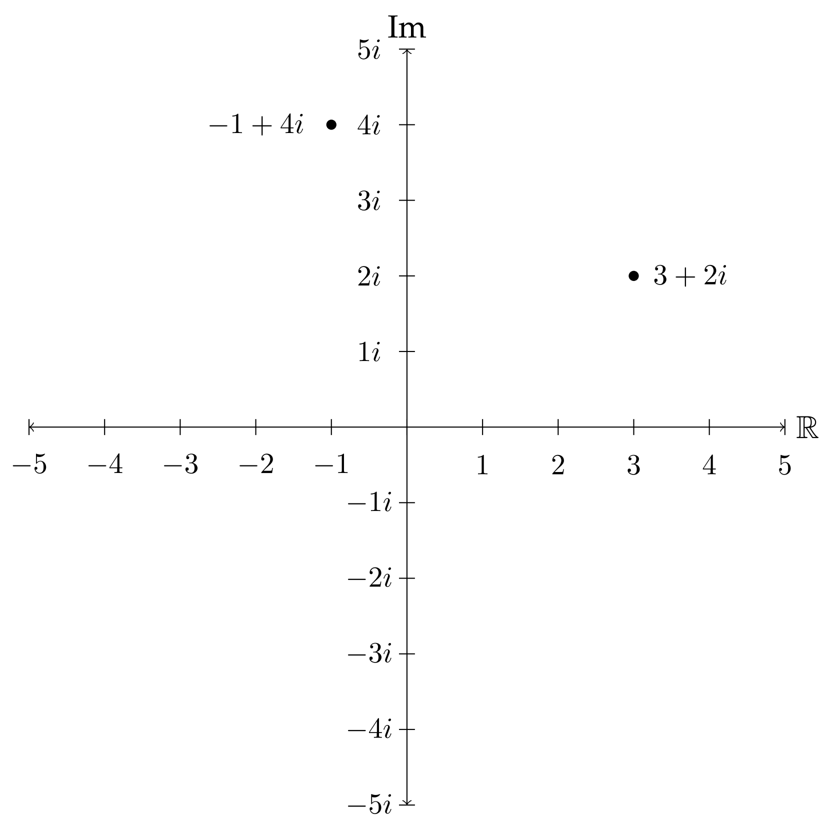
One of the best ways to conceptualize the Real number system is on the number line - every point on the number line corresponds to a unique Real number and every Real number corresponds to a unique position on the Real number line.



After the development of the printing press in the 15th century, Fibonacci's *Liber Abaci* was translated into Italian from Latin and read throughout Italy. As a result, Italy became a thriving center of mathematics until the 17th century, when the center of European mathematics moved north to France, Germany and England.

Throughout the 1500's Italian mathematicians such as Girolamo Cardano, Raphael Bombelli and Niccolo Fontana Tartaglia worked to extend the ideas in Fibonacci's book. They produced formulas to solve cubic (x^3), and quartic (x^4) degree equations. In solving some of these equations they found that their formulas sometimes produced negative values under a square root. None of the known number systems could accomodate this possibility. In Cardano's book on algebra *Ars Magna*, he encounters a problem which involves the square root of a negative number. He says, "It is clear that this case is impossible. Nevertheless we will work thus..." and he proceeds to compute a valid complex solution to the problem. Mathematicians eventually defined the complex unit $\sqrt{-1} = i$ and then devised a system in which all complex numbers are a combination of a Real part (a) and an "imaginary" part (bi).

The complex numbers are a two dimensional number system represented by the double barred C: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$, where i is the complex unit defined as $i = \sqrt{-1}$. Throughout the late 1700's and early 1800's mathematicians gradually moved towards a geometrical interpretation of the two-dimensional complex numbers. What is today known as the "Argand diagram" represents the real valued portion of a complex number along a horizontal axis and the multiple of the complex unit along the vertical axis.



Expressing square roots of negative numbers

Square roots of negative quantities are generally expressed as a multiple of i.

Examples

$$\sqrt{-4} = 2i.$$

$$\sqrt{-25} = 5i$$

$$\sqrt{-7} \approx 2.646i$$

Adding, subtracting and multiplying with complex numbers

Calculating with complex numbers has many similarities to working with variables. The real part and imginary part are treated separately for addition and subtraction, but can be multiplied and divided.

Examples

Compute the following:

$$(6 - 4i) + (-2 + 7i) = 4 + 3i$$

$$(-9 + 2i) - (-4 + 6i) = -9 + 2i + 4 - 6i = -5 - 4i$$

$$3(10 + i) = 30 + 3i$$

$$-7i(-5 + 8i) = 35i - 56i^2$$

Now we encounter an interesting fact about complex numbers and, in particular, the complex unit i. By definition, $i = \sqrt{-1}$. Therefore, if we square i we should get -1. In the last example problem above, we can replace the i^2 with -1 to finish the problem.

$$\begin{aligned} -7i(-5+8i) &= 35i - 56i^2 \\ &= 35i - 56(-1) \\ &= 35i + 56 \\ &= 56 + 35i \end{aligned}$$

Examples

Compute the following:

$$\begin{aligned} (8-5i)(1-4i) &= 8 - 32i - 5i + 20i^2 \\ &= 8 - 37i + 20(-1) \\ &= 8 - 37i - 20 \\ &= -12 - 37i \end{aligned}$$

$$\begin{aligned} (9+2i)^2 &= (9+2i)(9+2i) \\ &= 81 + 18i + 18i + 4i^2 \\ &= 81 + 36i + 4(-1) \\ &= 81 + 36i - 4 \\ &= 77 + 36i \end{aligned}$$

Powers of i

The powers of i follow an interesting pattern based on the definition that $i^2 = -1$.

We can see that $i^1 = i$ and that $i^2 = -1$, as a result, $i^3 = i^2 * i^1 = -1 * i = -i$.

In a similar fashion, $i^4 = i^2 * i^2 = (-1)(-1) = 1$.

This means that $i^5 = i^4 * i = 1 * i = i$.

If we put all of this information together we get the following:

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i^1 = i$

$i^6 = i^2 = -1$

$i^7 = i^3 = -i$

$i^8 = i^4 = 1$

In other words, every power of i is equivalent to either i, -1, $-i$, or 1. To determine which of these values a power of i is equivalent to, we need to find the remainder of the exponent when it is divided by 4.

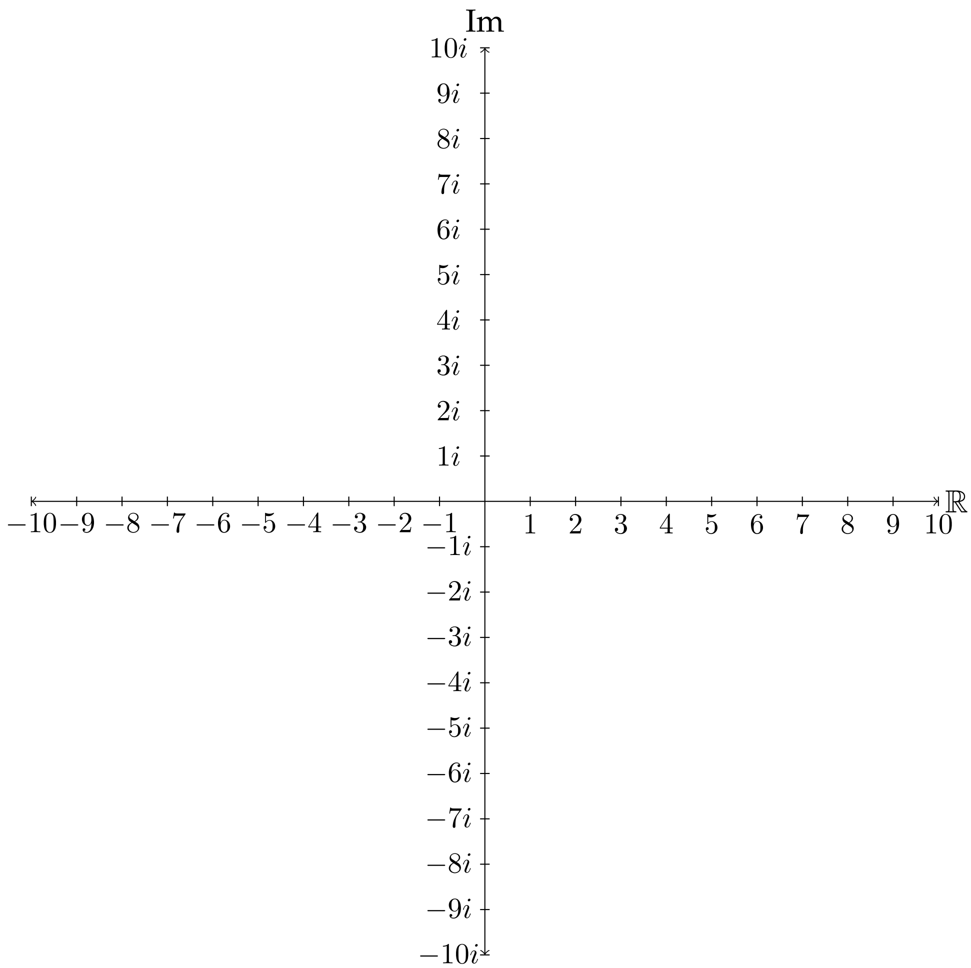
Example

Simplify i^{38}

Since every $i^4 = 1$, then $i^{38} = i^{36} * i^2 = (i^4)^9 * i^2 = 1^9 * i^2 = i^2 = -1$

Since 38 is 2 more than a multiple of 4, then $i^{38} = i^2 = -1$.

Exercises 1.4



Graph the following complex numbers.

1) $2+5i$ 2) $4-3i$ 3) $-2+6i$ 4) $-3-5i$

5) 4 6) $-2i$ 7) $7-i$ 8) $-1+i$

9) $-8+4i$ 10) $8+3i$ 11) $7i$ 12) $-5-9i$

Express each quantity in terms of i. Round irrational values to the nearest 1000th.

13) $\sqrt{-36}$ 14) $\sqrt{-81}$ 15) $\sqrt{-100}$ 16) $\sqrt{-49}$

17) $\sqrt{-4}$ 18) $\sqrt{-25}$ 19) $\sqrt{-2}$ 20) $\sqrt{-6}$

21) $\sqrt{-10}$ 22) $\sqrt{-31}$ 23) $\sqrt{-5}$ 24) $\sqrt{-3}$

Perform the indicated operation and simplify.

25) $(6+7i)+(5+3i)$

26) $(4-5i)+(3+9i)$

27) $(9+8i)-(1-2i)$

28) $(2+i)-(6-4i)$

29) $(7-4i)-(5-3i)$

30) $(8+i)-(4+3i)$

31) $(7i)(6i)$

32) $(4i)(-8i)$

33) $(-2i)(5i)$

34) $(12i)(3i)$

35) $(1+i)(3+2i)$

36) $(1+5i)(4+3i)$

37) $(6-5i)(2-3i)$

38) $(8-3i)(2+i)$

39) $(-3+4i)(-1-2i)$

40) $(-7-i)(3-5i)$

41) $(4-2i)^2$

42) $(-5+i)^2$

43) $(3+i)(3-i)$

44) $(2+6i)(2-6i)$

45) $(9-4i)(9+4i)$

46) $(5+2i)(5-2i)$

Express as either $i, -1, -i$, or 1.

47) i^3

48) i^7

49) i^{21}

50) i^{13}

51) i^{29}

52) i^{56}

53) i^{72}

54) i^{35}

55) i^{66}

56) i^{103}

57) i^{16}

58) i^{53}

59) i^{11}

60) i^{42}

61) i^{70}

62) i^9

1.5 Quadratic Equations with Complex Roots

In Section 1.3, we considered the solution of quadratic equations that had two real-valued roots. This was due to the fact that in calculating the roots for each equation, the portion of the quadratic formula that is square rooted ($b^2 - 4ac$, often called the *discriminant*) was always a positive number.

For example, in using the quadratic formula to calculate the the roots of the equation $x^2 - 6x + 3 = 0$, the discriminant is positive and we will end up with two real-valued roots:

$$x^2 - 6x + 3 = 0$$

$$a = 1, b = -6, c = 3$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2 * 1}$$

$$= \frac{6 \pm \sqrt{36 - 12}}{2}$$

$$= \frac{6 \pm \sqrt{24}}{2}$$

$$\approx \frac{6 \pm 4.899}{2}$$

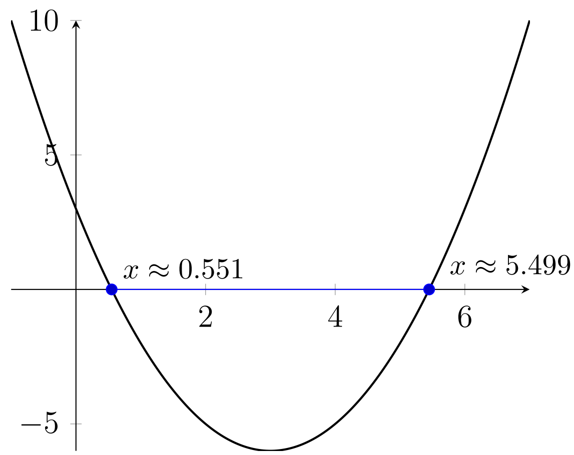
$$\approx \frac{6 + 4.899}{2} \qquad \approx \frac{6 - 4.899}{2}$$

$$\approx \frac{10.899}{2} \qquad \approx \frac{1.101}{2}$$

$$\approx 5.449 \qquad \approx 0.551$$

When we added and subtracted the square root of 24 to 6 in the quadratic formula, this created two answers, and they were real-valued because the square root of 24 is real-valued.

Another way to see this is graphically. If we graph $y = x^2 - 6x + 3$ and find the x values that make $y = 0$, these will appear along the x-axis, and will be the same values that solve the equation $x^2 - 6x + 3 = 0$.



If we consider a related, but slightly different equation to start with, these relationships between the roots, the discriminant and the graphical intersections will be slightly different.

$$x^2 - 6x + 9 = 0$$

$$a = 1, b = -6, c = 9$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(9)}}{2 * 1}$$

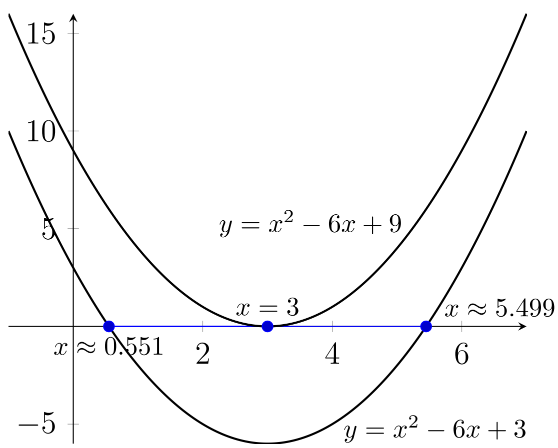
$$= \frac{6 \pm \sqrt{36 - 36}}{2}$$

$$= \frac{6 \pm \sqrt{0}}{2}$$

$$= \frac{6}{2} = 3$$

Because the discriminant was 0 in this problem, we only get one real-valued answer.

Graphically, the additional 6 that was added to the original equation to change it from $x^2 - 6x + 3$ to $x^2 - 6x + 9$ shifts every y value on the graph up 6 units.



If we add an additional three units to the constant term of this quadratic equation, we encounter a third possibility.

$$x^2 - 6x + 12 = 0$$

$$a = 1, b = -6, c = 12$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(12)}}{2 * 1}$$

$$= \frac{6 \pm \sqrt{36 - 48}}{2}$$

$$= \frac{6 \pm \sqrt{-12}}{2}$$

$$= \frac{6 \pm i\sqrt{12}}{2}$$

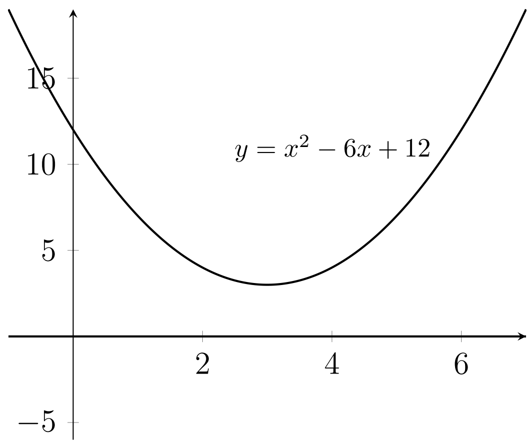
$$\approx \frac{6 \pm 3.464i}{2}$$

$$= \frac{6}{2} \pm \frac{3.464i}{2}$$

$$\approx 3 \pm 1.732i$$

Here the discriminant is negative, which leads to two complex-valued answers. If the equation has real-valued coefficients, the complex roots will always come in *conjugate* pairs. Complex conjugates share the same real-valued part and have opposite signs in their complex-valued (or imaginary) parts: $a \pm bi$

Graphically, the previous problem was one step away from not intersecting the x-axis at all and the additional three units that we added on to get $y = x^2 - 6x + 12$ moves the graph entirely away from the x-axis. Because the roots are complex-valued, we don't see any roots on the x-axis. The x-axis contains only real numbers.



Since the calculator has been programmed for the quadratic formula, the focus of the problems in this section will be on putting them into standard form.

Example

Solve for x.

$$(2x+1)(x+5) - 2x(x+7) = 5(x+3)^2$$

$$2x^2 + 11x + 5 - 2x^2 - 14x = 5(x+3)(x+3)$$

$$-3x + 5 = 5(x^2 + 6x + 9)$$

$$-3x + 5 = 5x^2 + 30x + 45$$

$$0 = 5x^2 + 33x + 40$$

$$a = 5, b = 33, c = 40$$

$$x = -5, -1.6$$

The fact that the roots of this equation were rational numbers means that the equation could have been solved by factoring.

$$0 = 5x^2 + 33x + 40$$

$$0 = (5x + 8)(x + 5)$$

$$5x + 8 = 0 \qquad x + 5 = 0$$

$$5x = -8 \qquad x = -5$$

$$x = -1.6$$

Example

Solve for x.

$$(x - 2)^2 + 3(4x - 1)(x + 1) = 7(x + 1)(x - 1)$$

$$x^2 - 4x + 4 + 3(4x^2 + 3x - 1) = 7(x^2 - 1)$$

$$x^2 - 4x + 4 + 12x^2 + 9x - 3 = 7x^2 - 7$$

$$13x^2 + 5x + 1 = 7x^2 - 7$$

$$6x^2 + 5x + 8 = 0$$

$$a = 6, b = 5, c = 8$$

$$x \approx -0.41\overline{6} \pm 1.077i \approx -\tfrac{5}{12} \pm 1.077i$$

Exercises 1.5

Solve for x in each equation. Round any irrational values to the nearest 1000th.

1) $3x^2 - 3x = 4$

2) $4x^2 - 2x = 7$

3) $5x^2 = 3 - 7x$

4) $3x^2 = 21 = 14x$

5) $6x^2 + 1 = 2x$

6) $5x - 3x^2 = 17$

7) $(5x - 1)(2x + 3) = 3x - 20$

8) $(x + 4)(3x - 1) = 9x - 5$

9) $(x - 2)^2 = 8x(x - 1) + 10$

10) $(2x - 3)^2 = 2x - 7x^2$

11) $(x + 5)(x - 6) = (2x - 1)(x - 4)$

12) $(3x - 4)(x + 2) = (2x - 5)(x + 5)$

1.6 Multiplying and Dividing Rational Expressions

Reducing Rational Expressions

A *rational expression* is simply an algebraic fraction, and our first consideration will be to reduce these expressions to lowest terms in the same way that we reduce numerical fractions to lowest terms. When we reduce $\frac{6}{15}$ to $\frac{2}{5}$ by canceling the common factor of three, we are removing a redundant factor of 1 in the form of $\frac{3}{3}$.

$$\begin{aligned}\frac{6}{15} = \frac{3*2}{3*5} &= \frac{3}{3}*\frac{2}{5} \\ &= 1*\frac{2}{5} \\ &= \frac{2}{5}\end{aligned}$$

Similarly, if there is a common factor that can be factored out of an algebraic fraction, this also can be canceled.

$$\begin{aligned}\frac{21x+14}{7x+7} &= \frac{7(3x+2)}{7(x+1)} \\ &= \frac{7}{7}*\frac{3x+2}{x+1} \\ &= 1*\frac{3x+2}{x+1} \\ &= \frac{3x+2}{x+1}\end{aligned}$$

It's important to remeber that only common factors can be canceled. This means that the first priority in each problem will be to identify the factors of the numerator and denominator to see if they share any common factors.

Example

Reduce to lowest terms.

$$\frac{x^2+4x-12}{x^2-4}$$

$$\begin{aligned}\frac{x^2+4x-12}{x^2-4} &= \frac{(x+6)(x-2)}{(x+2)(x-2)} \\ &= \frac{(x+6)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} \\ &= \frac{x+6}{x+2}\end{aligned}$$

Notice that we can't cancel the 6 and the 2 in the final answer because they aren't factors. The plus signs in the numerator and denominator prevent us from cancelling the 6 and the 2.

In the previous examples, we saw that cancelling out common factors in the numerator and denominator was actually a process of eliminating a redundant factor of 1. In the following example, we'll see a slightly different form of cancelling.

Example

Reduce to lowest terms.

$$\frac{16-x^2}{x^2+x-20}$$

$$\frac{16-x^2}{x^2+x-20} = \frac{(4+x)(4-x)}{(x+5)(x-4)}$$

In this problem, there are no common factors, but we can do some cancelling. We can see that $(4-x)$ and $(x-4)$ are not the same expression. In the first binomial, the 4 is positive and the x is negative, whereas in the second binomial, the 4 is negative and the x is positive. So, we know that $\frac{4-x}{x-4} \neq 1$. However, if we factor a (-1) out of the numerator, we will see an interesting phenomenon:

$$\frac{4-x}{x-4} = \frac{-1(-4+x)}{x-4}$$

$$= \frac{-1(x-4)}{x-4}$$

$$= -1 * \frac{x-4}{x-4}$$

$$= -1 * 1 = -1$$

Therefore, although $\frac{4-x}{x-4} \neq 1$, we can say that $\frac{4-x}{x-4} = -1$. This will allow us to cancel $(4-x)$ and $(x-4)$ and replace them with (-1).

$$\frac{16-x^2}{x^2+x-20} = \frac{(4+x)(4-x)}{(x+5)(x-4)}$$

$$= \frac{(4+x)\cancel{(4-x)}}{(x+5)\cancel{(x-4)}(-1)}$$

In the final answer, the (-1) can be placed in the denominator or the numerator, but not both. It can also be placed in front of the fraction.

$$\frac{4+x}{-1(x+5)} = \frac{-1(4+x)}{x+5}$$

$$= -\frac{4+x}{x+5}$$

Multiplying and Dividing Rational Expressions

In multiplying and dividing rational expressions, it is often easier to identify and cancel out common factors before multiplying rather than afterwards. Multiplying rational expressions works the same way that multiplying numerical fractions does - multiply straight across the top and straight across the bottom. As a result, any factor in either numerator of the problem will end up in the numerator of the answer. Likewise, any factor in either denominator of the problem will end up in the denominator of the answer. Thus, any factor in either numerator can be cancelled with any factor in either denominator.

Example

Multiply. Express your answer in simplest form.

$\frac{x^2+5x+6}{25-x^2} * \frac{x^2-2x-15}{x^2+6x+9}$

$$\frac{x^2+5x+6}{25-x^2} * \frac{x^2-2x-15}{x^2+6x+9} = \frac{(x+2)(x+3)}{(5+x)(5-x)} * \frac{(x-5)(x+3)}{(x+3)(x+3)}$$

$$= \frac{(x+2)\cancel{(x+3)}}{(5+x)\cancel{(5-x)}(-1)} * \frac{\cancel{(x-5)}\cancel{(x+3)}}{\cancel{(x+3)}\cancel{(x+3)}}$$

$$= -\frac{x+2}{x+5}$$

Dividing rational expressions works in much the same way that dividing numerical fractions does. We multiply by the reciprocal. There are several ways to demonstrate that this is a valid definition for dividing. First, it is important to understand that the fraction bar is the same as a "divided by" symbol:

$$\frac{8}{2} = 8 \div 2 = 4.$$

The same is true for dividing fractions:

$$\frac{1}{3} \div \frac{2}{5} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}.$$

We can take the complex fraction $\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}$ and multiply it by 1 without changing its value:

$$\frac{1}{3} \div \frac{2}{5} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} * 1$$

$$= \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}$$

We can multiply by any form of 1 we want to and not change the value of the result.

$$\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} * 1 = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}$$

$$\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} * \frac{9}{9} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}$$

$$\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} * \frac{12}{12} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}$$

With a carefully chosen form of 1, we can transform the division problem into a multiplication problem.

$$\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} * \frac{\left(\frac{5}{2}\right)}{\left(\frac{5}{2}\right)} = \frac{\frac{1}{3} * \frac{5}{2}}{\frac{2}{5} * \frac{5}{2}}$$

$$= \frac{\frac{1}{3} * \frac{5}{2}}{1}$$

$$= \frac{1}{3} * \frac{5}{2}$$

$$= \frac{5}{6}$$

In this way, we can redefine division as multiplication by a reciprocal.

Example

Divide the expressions. Express your answer in lowest form.

$$\frac{2x^2 - x - 3}{x^2 - x - 12} \div \frac{x^2 + 5x + 4}{16 - x^2}$$

$$\frac{2x^2 - x - 3}{x^2 - x - 12} \div \frac{x^2 + 5x + 4}{16 - x^2} = \frac{2x^2 - x - 3}{x^2 - x - 12} * \frac{16 - x^2}{x^2 + 5x + 4}$$

$$= \frac{(2x - 3)(x + 1)}{(x - 4)(x + 3)} * \frac{(4 + x)(4 - x)}{(x + 1)(x + 4)}$$

$$= \frac{(2x - 3)\cancel{(x + 1)}}{\cancel{(x - 4)}(x + 3)} * \frac{\cancel{(4 + x)}\cancel{(4 - x)}(-1)}{\cancel{(x + 1)}\cancel{(x + 4)}}$$

$$= -\frac{2x - 3}{x + 3}$$

Exercises 1.6

Reduce each expression to lowest terms.

1) $\dfrac{3x+9}{x^2-9}$

2) $\dfrac{4x^2+8x}{12x+24}$

3) $\dfrac{x^2-2x}{6-3x}$

4) $\dfrac{15x^2+24x}{3x^2}$

5) $\dfrac{24x^2}{12x^2-6x}$

6) $\dfrac{x^2+4x+4}{x^2-4}$

7) $\dfrac{25-y^2}{2y^2-8y-10}$

8) $\dfrac{3y^2-y-2}{3y^2+5y+2}$

9) $\dfrac{x^2+4x-5}{x^2-2x+1}$

10) $\dfrac{x-x^2}{x^2+x-2}$

11) $\dfrac{x^2+5x-14}{2-x}$

12) $\dfrac{2x^2+5x-3}{1-2x}$

Multiply or divide the expressions in each problem.

Express your answers in lowest terms.

13) $\dfrac{3x+6}{5x^2} * \dfrac{x}{x^2-4}$

14) $\dfrac{4x^2}{x^2-16} * \dfrac{7x-28}{6x}$

15) $\dfrac{2a^2-7a+6}{4a^2-9} * \dfrac{4a^2+12a+9}{a^2-a-2}$

16) $\dfrac{4a^2-4a-3}{8a+4a^2} * \dfrac{16a^2}{4a^2-6a}$

17) $\dfrac{x^2-y^2}{(x+y)^3} * \dfrac{(x+y)^2}{(x-y)^2}$

18) $\dfrac{2x^2+x-3}{x^2-1} * \dfrac{2x-2}{2x^2+5x+3}$

19) $\dfrac{6x}{x^2-4} \div \dfrac{3x-9}{2x+4}$

20) $\dfrac{12x}{5x+20} \div \dfrac{4x^2}{x^2-16}$

21) $\dfrac{9x^2+3x-2}{6x^2-2x} \div \dfrac{3x+2}{6x^2}$

22) $\dfrac{2a^2-5a-3}{4a^2+2a} \div \dfrac{2a+1}{4a}$

23) $\dfrac{x^2+7x+6}{x^2+x-6} \div \dfrac{x^2+5x-6}{x^2+5x+6}$

24) $\dfrac{x^2+7x+10}{x^2-x-30} \div \dfrac{3x^2+7x+2}{9x^2-1}$

25) $\dfrac{2x^2-x-28}{3x^2-x-2} \div \dfrac{4x^2+16x+7}{3x^2+11x+6}$

26) $\dfrac{9x^2+3x-2}{12x^2+5x-2} \div \dfrac{9x^2-6x+1}{8x^2-10x-3}$

1.7 Adding and Subtracting Rational Expressions

Just as we do with numerical fractions, we will need to have common denominators in order to add or subtract algebraic fractions. When we add $\frac{1}{2}+\frac{1}{3}$, we make a common denominator of 6 so that we can add them together.

$$\frac{1}{2}+\frac{1}{3}=\frac{1}{2}*\frac{3}{3}+\frac{1}{3}*\frac{2}{2}$$

$$=\frac{3}{6}+\frac{2}{6}$$

$$=\frac{5}{6}$$

Because the denominators, 2 and 3, are prime and don't share any common factors, the common denominator is simply $3*2=6$. We can see a similar result in adding algebraic fractions.

Example

Add the fractions. Express your answer in lowest terms.

$\frac{2}{x}+\frac{x}{x-3}$

$$\frac{2}{x}+\frac{x}{x-3}=\frac{2}{x}*\frac{x-3}{x-3}+\frac{x}{x-3}*\frac{x}{x}$$

$$=\frac{2(x-3)}{x(x-3)}+\frac{x*x}{x(x-3)}$$

$$=\frac{2x-6+x^2}{x(x-3)}=\frac{x^2+2x-6}{x(x-3)}$$

It's important to be aware that in subtraction, the negative sign representing subtraction must be distributed to all terms in the second numerator.

Example

Subtract the given expressions. Express your answer in lowest terms.

$\frac{6}{x+1} - \frac{x+5}{x-2}$

$$\begin{aligned}
\frac{6}{x+1} - \frac{x+5}{x-2} &= \frac{6}{x+1} * \frac{x-2}{x-2} - \frac{x+5}{x-2} * \frac{x+1}{x+1} \\
&= \frac{6(x-2) - (x+5)(x+1)}{(x+1)(x-2)} \\
&= \frac{6x - 12 - (x^2 + 6x + 5)}{(x+1)(x-2)} \\
&= \frac{6x - 12 - x^2 - 6x - 5}{(x+1)(x-2)} \\
&= \frac{-x^2 - 17}{(x+1)(x-2)}
\end{aligned}$$

In other situations, the denominators may share a common factor. In this case, we can turn one of the denominators into the other one:

Example

Add the given fractions.

$\frac{7}{x^2 + 8x + 15} + \frac{2}{x+3}$

$$\frac{7}{x^2+8x+15}+\frac{2}{x+3}=\frac{7}{(x+3)(x+5)}+\frac{2}{x+3}$$

We can turn $(x+3)$ into $x^2+8x+15$ by multiplying by $(x+5)$

$$\frac{7}{(x+3)(x+5)}+\frac{2}{x+3}=\frac{7}{(x+3)(x+5)}+\frac{2}{x+3}*\frac{x+5}{x+5}$$

$$=\frac{7}{(x+3)(x+5)}+\frac{2(x+5)}{(x+3)(x+5)}$$

$$=\frac{7+2x+10}{(x+3)(x+5)}$$

$$=\frac{2x+17}{(x+3)(x+5)}$$

Sometimes, the answer we end up with is not in lowest terms:

Example

Add the fractions.

$$\frac{x}{x+2}+\frac{8}{x^2+8x+12}$$

$$\frac{x}{x+2}+\frac{8}{x^2+8x+12}=\frac{x}{x+2}+\frac{8}{(x+2)(x+6)}$$

$$=\frac{x}{x+2}*\frac{x+6}{x+6}+\frac{8}{(x+2)(x+6)}$$

$$=\frac{x(x+6)}{(x+2)(x+6)}+\frac{8}{(x+2)(x+6)}=\frac{x(x+6)+8}{(x+2)(x+6)}$$

$$\frac{x(x+6)+8}{(x+2)(x+6)} = \frac{x^2+6x+8}{(x+2)(x+6)}$$

The numerator is factorable:

$$\begin{aligned}\frac{x^2+6x+8}{(x+2)(x+6)} &= \frac{(x+2)(x+4)}{(x+2)(x+6)} \\ &= \frac{\cancel{(x+2)}(x+4)}{\cancel{(x+2)}(x+6)} \\ &= \frac{x+4}{x+6}\end{aligned}$$

Exercises 1.7

Add or subtract the given expressions.

1) $\dfrac{1}{x-1} - \dfrac{1}{x}$

2) $\dfrac{3}{y-6} - \dfrac{1}{y}$

3) $\dfrac{2}{x-3} + \dfrac{4}{x+3}$

4) $\dfrac{3}{x+4} - \dfrac{4}{x-2}$

5) $\dfrac{3}{k+2} - \dfrac{k-4}{k+5}$

6) $\dfrac{a+1}{a} - \dfrac{a}{a+1}$

7) $\dfrac{2y}{y^2-25} - \dfrac{y}{y-5}$

8) $\dfrac{x}{x^2-1} + \dfrac{4}{x+1}$

9) $\dfrac{1}{x-3} + \dfrac{x}{x+1}$

10) $\dfrac{9y}{y-4} - \dfrac{y+1}{y+5}$

Add or subtract the given expressions. Express your answers in lowest terms.

11) $\dfrac{b}{b+1} - \dfrac{b+1}{2b+2}$

12) $\dfrac{4x+1}{8x-12} + \dfrac{x-3}{2x-3}$

13) $\dfrac{2}{a^2+4a+3} + \dfrac{1}{a+3}$

14) $\dfrac{1}{y+6} - \dfrac{4}{y^2+8y+12}$

15) $\dfrac{x+1}{2x+4} - \dfrac{x^2}{2x^2-8}$

16) $\dfrac{x+1}{x+2} - \dfrac{x^2+1}{x^2-x-6}$

17) $\dfrac{2x}{x^2 - 3x + 2} + \dfrac{2x}{x - 1} - \dfrac{x}{x - 2}$

18) $\dfrac{3x + 3}{2x^2 - x - 1} + \dfrac{1}{2x + 1}$

19) $\dfrac{4a}{a - 2} - \dfrac{3a}{a - 3} + \dfrac{4a}{a^2 - 5a + 6}$

20) $\dfrac{2}{y - 3} - \dfrac{8 - 4y}{y^2 - 8y + 15}$

21) $\dfrac{2x}{x - 1} + \dfrac{3x}{x + 1} - \dfrac{x + 3}{x^2 - 1}$

22) $\dfrac{a}{a - 1} - \dfrac{2}{a + 2} + \dfrac{3(a - 2)}{a^2 + a - 2}$

23) $\dfrac{x}{x - 1} + \dfrac{x + 7}{x^2 - 1} - \dfrac{x - 2}{x + 1}$

24) $\dfrac{2y + 5}{y^2 - 16} - \dfrac{y - 9}{y^2 - y - 12}$

1.8 Complex Fractions

Complex fractions involve simplifying a rational expression which has a complicated numerator and/or denominator.

Example

Simplify.

$$\frac{3+\frac{x}{x+2}}{1-\frac{x+3}{x-1}}$$

There are a variety of ways to approach this problem. One of the most straightforward ways to simplify the expression above is to create common denominators for the numerator and the denominator so that each one is a single fractional expression:

$$\frac{3+\frac{x}{x+2}}{1-\frac{x+3}{x-1}} = \frac{\frac{3}{1}*\frac{x+2}{x+2}+\frac{x}{x+2}}{\frac{1}{1}*\frac{x-1}{x-1}-\frac{x+3}{x-1}}$$

$$= \frac{\left(\frac{3x+6+x}{x+2}\right)}{\left(\frac{x-1-(x+3)}{x-1}\right)}$$

$$= \frac{\left(\frac{4x+6}{x+2}\right)}{\left(\frac{-4}{x-1}\right)} \qquad \text{(Now this is a division problem)}$$

$$= \frac{4x+6}{x+2}*\frac{x-1}{-4} = \frac{2(2x+3)}{x+2}*\frac{x-1}{-4}$$

$$= \frac{\cancel{2}(2x+3)}{x+2}*\frac{x-1}{\cancel{-4}(-2)} = \frac{(2x+3)(x-1)}{-2(x+2)}$$

Simplifying complex fractions uses all of the previous concepts about rational expressions which we've covered in this chapter.

Example

Simplify.

$$\frac{x - \frac{x}{x+3}}{1 + \frac{2}{x}}$$

$$\frac{x - \frac{x}{x+3}}{1 + \frac{2}{x}} = \frac{\frac{x}{1} * \frac{x+3}{x+3} - \frac{x}{x+3}}{\frac{1}{1} * \frac{x}{x} + \frac{2}{x}} \quad \text{creating common denominators}$$

$$= \frac{\left(\frac{x(x+3) - x}{x+3}\right)}{\left(\frac{x+2}{x}\right)}$$

$$= \frac{\left(\frac{x^2 + 3x - x}{x+3}\right)}{\left(\frac{x+2}{x}\right)} = \frac{\left(\frac{x^2 + 2x}{x+3}\right)}{\left(\frac{x+2}{x}\right)} \quad \text{dividing fractions}$$

$$= \frac{x^2 + 2x}{x+3} * \frac{x}{x+2} = \frac{x(x+2)}{x+3} * \frac{x}{x+2}$$

$$= \frac{x\cancel{(x+2)}}{x+3} * \frac{x}{\cancel{x+2}} \quad \text{factor and cancel to reduce to lowest terms}$$

$$= \frac{x^2}{x+3}$$

Exercises 1.8

Simplify each complex fraction. Express your answer in lowest terms.

1) $$\frac{1}{\left(x+\frac{y}{2}\right)}$$

2) $$\frac{\left(\frac{1}{x}+\frac{1}{y}\right)}{\left(\frac{y}{x}-\frac{x}{y}\right)}$$

3) $$\frac{\left(1+\frac{m}{n}\right)}{\left(1-\frac{n^2}{m^2}\right)}$$

4) $$\frac{\left(\frac{1}{x}-\frac{1}{y}\right)}{\left(\frac{1}{x^2}-\frac{1}{y^2}\right)}$$

5) $$\frac{\left(\frac{x}{y}-\frac{x-y}{x+y}\right)}{\left(\frac{y}{x}+\frac{x-y}{x+y}\right)}$$

6) $$\frac{\left(\frac{7}{a+1}-\frac{3}{a}\right)}{\left(\frac{3}{a}+\frac{1}{a-1}\right)}$$

7) $$\frac{\left(x-\frac{1}{2x+1}\right)}{\left(1-\frac{2}{2x+1}\right)}$$

8) $$\frac{\left(\frac{1}{2x-2}-\frac{1}{x}\right)}{\left(\frac{2}{x}-\frac{1}{x-1}\right)}$$

9) $$\frac{\left(x+\frac{4}{x+4}\right)}{\left(x-\frac{4x+4}{x+4}\right)}$$

10) $$\frac{\left(x-\frac{x+6}{x+2}\right)}{\left(x-\frac{4x+15}{x+2}\right)}$$

11) $$\frac{\left(\frac{1}{x+2}-\frac{1}{x-3}\right)}{\left(1+\frac{1}{x^2-x-6}\right)}$$

12) $$\frac{\left(1-\frac{1}{x+1}\right)}{\left(1+\frac{1}{x-1}\right)}$$

13) $$\frac{\left(\frac{1}{a-b}-\frac{3}{a+b}\right)}{\left(\frac{2}{b-a}+\frac{4}{b+a}\right)}$$

14) $$\frac{\left(\frac{3}{y^2-4}\right)}{\left(\frac{1}{y+2}-\frac{1}{y-2}\right)}$$

15) $$\frac{\left(n+2-\frac{5}{n-2}\right)}{\left(1-\frac{1}{(n-2)^2}\right)}$$

16) $$\frac{\left(4+\frac{1}{x+1}\right)}{\left(16-\frac{1}{(x+1)^2}\right)}$$

17) $$\frac{\left(2+\frac{x-2}{1-x^2}\right)}{\left(2-\frac{3}{x+1}\right)}$$

18) $$\frac{\left(\frac{1}{2x-1}-\frac{1}{2x+1}\right)}{\left(4-\frac{1}{x^2}\right)}$$

1.9 Rational Equations

We can also use the skills we have covered in previous sections to solve equations involving rational expressions. There are three main methods of solution that will be explored in this section - multiplying on both sides to clear a denominator, cross-multiplying, and making common denominators. Each of these techniques is actually the same process, but approached from a slightly different perspective.

Clearing a denominator

Often, if the denominator is a single variable, it can be easy and striaghtforward to multiply on both sides by the denominator to cancel it out.

Example

Solve for x.

$x + \frac{4}{x} = 7$

If we multiply on both sides by x, it will clear the variable from the denominator:

$$x + \frac{4}{x} = 7$$

$$x * \left(x + \frac{4}{x}\right) = (7) * x \qquad \text{Multiply on both sides by } x$$

$$x * x + x * \frac{4}{x} = 7x \qquad \text{Distribute the } x$$

$$x^2 + 4 = 7x$$

$$x^2 - 7x + 4 = 0 \qquad \text{Standard form}$$

$$x \approx 0.628, 6.372$$

If a problem is stated simply as the equality of two fractions, cross-multiplying can be a useful method of solution.

Example

Solve for x.

$\frac{x}{2x+3} = \frac{7}{x-4}$

$$\frac{x}{2x+3} = \frac{7}{x-4}$$

$$x(x-4) = 7(2x+3)$$

$$x^2 - 4x = 14x + 21$$

$$x^2 - 18x - 21 = 0$$

$$x \approx 19.100, -1.100$$

Cross-multiplying is really just a short-cut method of clearing out the denominators by multiplying on both sides by both denominators:

$$\frac{x}{2x+3} = \frac{7}{x-4}$$

$$(x-4)(2x+3) * \frac{x}{2x+3} = \frac{7}{x-4} * (x-4)(2x+3)$$

$$(x-4)\cancel{(2x+3)} * \frac{x}{\cancel{2x+3}} = \frac{7}{\cancel{x-4}} * \cancel{(x-4)}(2x+3)$$

$$x(x-4) = 7(2x+3)$$

Then the equation is ready to be solved as shown above - but by just cross-mulitiplying, we skip directly to the solution portion of the problem.

Sometimes, it is helpful to create a common denominator in order to set up a situation where cross-multiplying can be used.

Example

Solve for x.

$\frac{1}{x+6}+\frac{4}{x-2}=\frac{3}{x+1}$

$$\frac{1}{x+6}+\frac{4}{x-2}=\frac{3}{x+1}$$

$$\frac{1}{x+6}*\frac{x-2}{x-2}+\frac{4}{x-2}*\frac{x+6}{x+6}=\frac{3}{x+1}$$

$$\frac{1(x-2)+4(x+6)}{(x+6)(x-2)}=\frac{3}{x+1}$$

$$\frac{x-2+4x+24}{(x+6)(x-2)}=\frac{3}{x+1}$$

$$\frac{5x+22}{(x+6)(x-2)}=\frac{3}{x+1}$$

$$(5x+22)(x+1)=3(x+6)(x-2)=3(x^2+4x-12)$$

$$5x^2+27x+22=3x^2+12x-36$$

$$2x^2+15x+58=0$$

$$x\approx -3.75\pm 3.865i$$

Example

Solve for x.

$\frac{2}{x-2}+\frac{x}{2x-1}=4$

$$\frac{2}{x-2}+\frac{x}{2x-1}=4$$

$$\frac{2}{x-2}*\frac{2x-1}{2x-1}+\frac{x}{2x-1}*\frac{x-2}{x-2}=4$$

$$\frac{2(2x-1)+x(x-2)}{(x-2)(2x-1)}=4$$

$$\frac{4x-2+x^2-2x}{(x-2)(2x-1)}=4$$

$$\frac{x^2+2x-2}{(x-2)(2x-1)}=\frac{4}{1}$$

$$1(x^2-2x-2)=4(x-2)(2x-1)=4(2x^2-5x+2)$$

$$x^2-2x-2=8x^2-20x+8$$

$$0=7x^2-18x+10$$

$$x\approx 1.760, 0.812$$

In some situations, we can create a single common denominator for every fraction in the problem and then clear them all at once.

Example

Solve for x.

$\frac{2}{x+3} - \frac{4}{3x-1} = \frac{x}{3x^2+8x-3}$

$$\frac{2}{x+3} - \frac{4}{3x-1} = \frac{x}{3x^2+8x-3}$$

$$\frac{2}{x+3} * \frac{3x-1}{3x-1} - \frac{4}{3x-1} * \frac{x+3}{x+3} = \frac{x}{3x^2+8x-3}$$

$$\frac{2(3x-1)-4(x+3)}{(x+3)(3x-1)} = \frac{x}{(x+3)(3x-1)}$$

$$2(3x-1)-4(x+3) = x$$

The missing step above is the clearing of both denominators:

$$(x+3)(3x-1) * \frac{2(3x-1)-4(x+3)}{(x+3)(3x-1)} = \frac{x}{(x+3)(3x-1)} * (x+3)(3x-1)$$

$$\cancel{(x+3)(3x-1)} * \frac{2(3x-1)-4(x+3)}{\cancel{(x+3)(3x-1)}} = \frac{x}{\cancel{(x+3)(3x-1)}} * \cancel{(x+3)(3x-1)}$$

$$2(3x-1)-4(x+3) = x$$

As is true in the process of cross-multiplying, it isn't necessary to actually cancel out the denominators in completing the problem.

$$\frac{2}{x+3} - \frac{4}{3x-1} = \frac{x}{3x^2+8x-3}$$

$$\frac{2}{x+3} * \frac{3x-1}{3x-1} - \frac{4}{3x-1} * \frac{x+3}{x+3} = \frac{x}{3x^2+8x-3}$$

$$\frac{2(3x-1)-4(x+3)}{(x+3)(3x-1)} = \frac{x}{(x+3)(3x-1)}$$

$$2(3x-1)-4(x+3) = x$$

$$6x-2-4x-12 = x$$

$$2x-14 = x$$

$$x = 14$$

Exercises 1.9

1) $x+\frac{5}{x}=-6$

2) $x+\frac{6}{x}=-7$

3) $y-\frac{5}{y}=2$

4) $\frac{7}{a}+1=a$

5) $\frac{9}{2y+4}=\frac{3}{y}$

6) $\frac{4}{3n+7}=\frac{1}{2}$

7) $\frac{x}{x+3}=\frac{8}{x+6}$

8) $\frac{y-2}{2}=\frac{5}{y-5}$

9) $\frac{2}{n}=\frac{n}{5n+12}$

10) $\frac{x}{4-x}=\frac{2}{x}$

11) $\frac{5x}{14x+3}=\frac{1}{x}$

12) $\frac{a}{8a+3}=\frac{1}{3a}$

13) $\frac{9}{x-1}-\frac{2}{x+4}=\frac{1}{x+2}$

14) $\frac{1}{x-2}+\frac{4}{x+5}=\frac{1}{x-3}$

15) $\frac{5}{3x+2}+\frac{1}{x-1}=\frac{3}{x+2}$

16) $\frac{1}{y-2}-\frac{4}{2y+5}=\frac{6}{y-1}$

17) $\dfrac{5}{x+1} + \dfrac{1}{x+2} = 3$

18) $\dfrac{1}{2x-1} - \dfrac{2}{x+7} = 1$

19) $\dfrac{6}{y-4} - \dfrac{1}{y+2} = 3$

20) $\dfrac{10}{a+1} + \dfrac{3}{a-2} = 2$

21) $\dfrac{3a}{a^2-2a-15} - \dfrac{a}{a+3} = \dfrac{2a}{a-5}$

22) $\dfrac{u^2+2}{u^2+u-2} - \dfrac{3u}{u+2} = \dfrac{-2u-1}{u-1}$

23) $\dfrac{4}{2x-1} + \dfrac{2}{x+3} = \dfrac{5}{2x^2+5x-3}$

24) $\dfrac{5}{x+5} - \dfrac{2}{x^2+2x-15} = \dfrac{2}{x-3}$

25) $\dfrac{5}{y-2} - \dfrac{3}{2y-1} = \dfrac{4}{2y^2-5y+2}$

26) $$\frac{x+2}{x-1}+\frac{x+4}{x}=\frac{2x+1}{x^2-x}$$

27) $$\frac{x}{x+2}+\frac{x+1}{x^2-7x-18}=\frac{5}{x-9}$$

28) $$\frac{2a}{a+7}-\frac{a}{a+3}=\frac{5}{a^2+10a+21}$$

29) $$\frac{x-1}{2x+1}-\frac{2x-3}{x+3}=\frac{3}{2x^2+7x+3}$$

30) $$\frac{y}{y+4}+\frac{6}{y+1}=\frac{y^2+4}{y^2+5y+4}$$

Addendum - Word Problems

Following are a selection of word problems - some from ancient times, some from the Renaissance and Enlightenment, and some from the 19th and 20th century.

1) A teacher agreed to work 9 months for $562.50 and board. At the end of the term, on account of two months absence caused by illness, he received only $409.50 for his seven months work. If the teacher used all nine months of his board during the term, what was his board per month? (American 1892)

2) A servant is promised $100 plus a cloak as wages for a year. After seven months, he leaves and receives $20 plus the cloak. How much is the cloak worth? (Clavius, German 1608)

3) The sales tax on garments is $\frac{1}{20}$ of their value. A certain man buys 42 garments, paying in copper coins. Two garments and 10 copper coins are paid as tax. What is the price of a garment, O learned one? (Ancient India)

4) Two wine merchants enter Paris, one of them with 64 casks of wine, the other with 20 casks (all of the same value). Since they do not have enough money to pay the customs duties, the first pays 5 casks of wine and 40 francs, and the second pays 2 casks of wine and receives 40 francs change. What is the price of each cask of wine and what is the duty on each cask? (Chuquet, French 1484)

5) One of two men had 12 fish and the other had 13 fish, and all of the fish were of the same price. From the first man, a customs agent took away 1 fish and 12 denarii for payment. From the other man he took 2 fish and gave him back 7 denarii as change. Find the customs fee and the price of each fish.
(Fibonacci, Italian 1202)

6) Two traders transporting sheepskins approach their country's border. The first trader has 100 sheepskins and the border guard takes 10 sheepskins plus $25 as a tariff. The second trader has 42 sheepskins and for a tariff, the border guard takes 7 sheepskins but returns $14 change. What is the tariff per sheepskin and what is the value of each sheepskin?

7) Two people have a certain amount of money. The first says to the second, “If you give me 5 denarii, I will have 7 times what you have left.” The second says to the first, “If you give me 7 denarii, I will have 5 times what you have left.” How much money does each have? Round to the nearest 10th.
(Leonardo, Italian c. 1500)

8) Two different scenarios from Ancient Greece:

Two friends were walking. One said to the other, “If I had 10 more coins, I would have 3 times as much money as you.” The other said, “If I had 10 more coins, I would have five times as much as you.” How many coins does each have?

Two friends were walking. One said to the other, “If you give me 10 of your coins, I would have 3 times as much money as you.” The other said, “If you give me 10 of your coins, I would have five times as much as you.” How many coins does each have?

9) Andy and Betty together have \$6 less than Christine. If Betty gives \$5 to Andy, then Andy will have half as much as Christine. If, instead, Andy gives \$5 to Betty, then Andy will have one-third as much as Betty. How much does each person have to begin with?

10) Three friends (A, B and C) each have a certain amount of money. A says, “I have as much as B plus one-third as much as C.” B says, “I have as much as C plus one-third as much as A.” C says, “I have 10 more than one-third of B.” How much does each person have? (Ancient Greece)

11) On a test, 39 more pupils passed than failed. On the next test, 7 who passed the first test failed and one-third of those who failed the first test passed the second. As a result, 31 more passed the second test than failed it. What was the record of passing and failing on the first test?

12) At two stations, A and B, six miles apart on the railway, the prices of coal are \$20 per ton and \$24 per ton respectively. The rates of cartage of coal are \$2.00 per ton per mile from A and \$3.00 per ton per mile from B. At a certain customer's home, on the railroad from A to B, the cost of coal is the same whether delivered from A or B. Find the distance to this home from A.

13) There were three-fourths as many women as there were men on the train. At the next station six men and eight women got off the train, and twelve men and five women got on. There were then three-fifths as many women as men on the train. How many men and how many women were originally on the train?

14) If a theater could put 5 more seats in a row, it would need 20 rows less, but if each row had 3 fewer seats, it would take 20 rows more to seat the same number. How many people will it seat?

15) An audience of 450 people is seated in rows, with the same number of people in each row. It would take 5 rows less if 3 more people were seated in each row. In how many rows are the people seated?

16) If evergreens are planted 4 feet closer together it will take 44 more trees for a certain piece of road, but if they are planted 6 feet farther apart, it will take 44 fewer trees for the same length of road. How many miles is the piece of road? (use 5280 feet = 1 mile)

17) A movie theater owner found that by raising the price of each ticket by \$1.00, 200 fewer people attened and she broke even, but that if she lowered the price by \$1.00 per person, 550 people attended and she increased her receipts by \$1000. What is the usual rate per person?

18) The brine pipes in an 84-foot width artificial hockey rink are equally spaced. If the space between each pair of pipes were increased by 1 inch, then 84 fewer lengths of pipe would be needed. What is the distance between the pipes now?

19) A living room shelf is 36 inches long and contains a certain number of books of uniform width. If each book were one-half inch narrower, the shelf would hold six more books. How many books of the wider variety does it hold?

20) I am a brazen lion; my spouts are my two eyes, my mouth and flat of my right foot. My right eye fills a jar in two days, my left eye in three and my foot in four. My mouth is capable of filling it in six hours. Tell me how long all four together will take to fill it. (Ancient Greece)

21) A man wishes to have 500 rubii of grain ground. He goes to a mill that has five stones. The first stone grinds 7 rubii of grain in an hour, the second grinds 5 rubii in an hour, the third 4 rubii in an hour, the fourth grinds 3 rubii per hour and the fifth grinds 1 rubii per hour. In how long will the grain be ground and how much is done by each stone? (Clavius, German 1583)

22) If two men and three boys can plow an acre in one-sixth of a day, how long would it require three men and two boys to plow it?
(Edward Brooks, American 1873)

23) A cobbler can cut leather for ten pairs of shoes in one day. He can finish 5 pairs of shoes in one day. How many pairs of shoes can he both cut and finish in one day? (Ancient Egypt)

24) Four waterspouts are filling a tank. Of the four spouts, one can fill the tank in one day, the second takes two days, the third takes three days and the fourth takes four days. How long will it take all four spouts working together to fill the tank? (Ancient Greece)

25) If, in one day, a person can make 30 arrows or fletch [put feathers on] 20 arrows, how many arrows can this person both make and fletch in one day? (Ancient China)

26) One military horse and one ordinary horse can pull a load of 40 *dan*. Two ordinary horses and one inferior horse can pull the same load of 40 *dan* as can three inferior horses and one military horse. How much can each horse pull individually? (Ancient China)

27) A barrel of water has several holes in it. The first hole empties the full barrel in 3 days. The second hole empties the full barrel in 5 days. The third hole empties the full barrel in 20 hours and the last hole empties the full barrel in 12 hours. With all the holes open, how long will it take to empty the barrel?
(Levi ben Gershon, French 1321)

28) A certain lion could eat a sheep in 4 hours; a leopard could do so in 5 hours; and a bear in 6 hours. How many hours would it take for all three animals to devour a sheep if it were thrown in among them? (Fibonacci, Italian 1202)

29) Two ships are some distance apart, which journey the first can complete in 5 days and the other in 7 days - it is sought in how many days they will meet if they begin the journey at the same time. (Fibonacci, Italian 1202)

30) Sarah, alone, can paint the garage in 24 hours, her sister Jenny, alone, can paint the same garage in 12 hours. With the aid of their mother, the three together can paint the garage in 4 hours. How long would it take their mother, working alone, to paint the garage?

31) It required 75 workers 38 days to build an embankment to be used for flood control. Had 18 workers been removed to another job at the very start of operations, how much longer would it have taken to build the embankment?

32) Mark, alone, requires 6 hours to paint a fence; however, his younger brother, who alone could do it in 9 hours, helps him. If they start work at 8:30am, at what time should they finish the work?

33) A group decides to build a cabin together. The job can be done by 3 skilled workers in 4 days or by 5 amateurs in 6 days. How long would it take if they all work together?

34) If it requires 18 workers 50 days to build a piece of road, how many days sooner would it be done if 7 more workers were hired at the beginning of operations?

35) A contractor estimated that a certain piece of work would be done by 9 carpenters in 8 hours or by 16 amateurs in 9 hours. The contractor wishes to get the job done as quickly as possible and uses both professional carpenters and amateurs on the same job. Four carpenters and 4 amateurs begin work at 6am. Allowing 45 minuntes for lunch, at what time should they finish the job?

36) Mrs. Ellis alone can do a piece of work in 6 days. Her oldest daughter takes 2 days longer; her youngest daughter takes twice as long as her mother. How long will it take to complete the job if all three work together?

37) If 5 men and 2 boys work together, a piece of work can be completed in one day and if 3 men and 6 boys work together, it can be completed in one day. How long would it take a boy to do the work alone?

38) A coal company can fill a certain order from one mine in 3 weeks and from a second mine in 5 weeks. How many weeks would be required to fill the order if both mines were used?

39) If 25 skilled workers work for 8 days, they can complete the construction of a concrete dam; 12 skilled workers and 15 untrained workers together can complete the dam in 10 days. How long would it take an untrained worker alone to complete the work on the dam?

40) A and B working togehter can complete a piece of work in 30 days. After they have both worked 18 days, however, A leaves and B finishes the work alone in 20 more days. Find the time in which each can do the work alone.

41) A dump cart can haul enough gravel to fill a pit in 6 days. A truck can do the same work in 2 days. How long would it take two dump carts and one truck working together to fill the pit?

Chapter 2

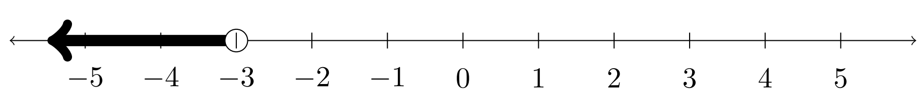
Polynomial and Rational Functions

This chapter will explore the solution of equations and inequalities involving both polynomial and rational functions, primarily through the examination of their graphical representations. We will also explore the use of polynomial long division and synthetic division in breaking down polynomials into their prime factors and the relationship between factors and roots.

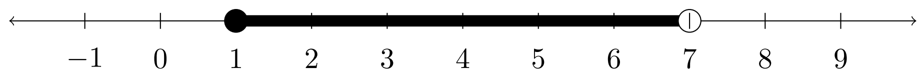
2.1 Representing Intervals

Many of the problems in this chapter will have solutions that must be expressed as an interval. This means a range of x values that will satisfy the original problem. In this section, we will introduce the translation of graphical intervals into algebraic notation.

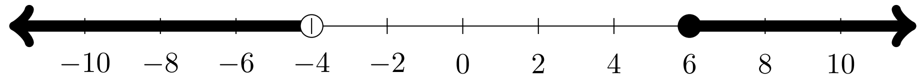
For example, in the diagram below, we would represent the interval shown on the graph as $x < -3$.



In this diagram, we would represent the interval shown on the graph as $1 \leq x < 7$.



In the next diagram, we have two separate intervals to represent - and we will need two separate statements to represent them.

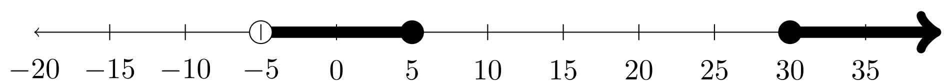


These intervals would be represented as $x < -4$ OR $x \geq 6$.

Sometimes students try to represent the intervals above as $6 \leq x < -4$, however, this expression would represent a single interval where x is both less than -4 and, *at the same time*, greater than or equal to $+6$. This is simply not possible, and would result in the empty set, which is the reason that the OR portion is needed in the correct answer.

Example

Represent the intervals indicated on the graph below:



On this graph, there is one interval beginning at -5 and ending at $+5$ and another beginning at 30 and continuing to infinity. Thus, these intervals would be represented as $-5 < x \leq 5$ OR $x \geq 30$.

Students familiar with another form of interval notation may wish to represent this interval as $(-5, 5] \cup [30, \infty)$. Both forms of notation accomplish the same goal.

Exercises 2.1

In each problem below, represent the intervals indicated on the graph.

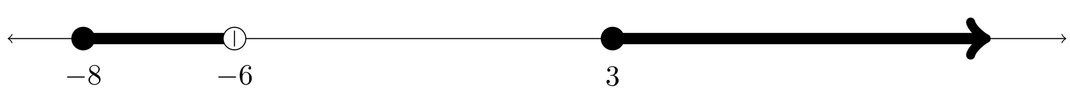
1)

2)

3)

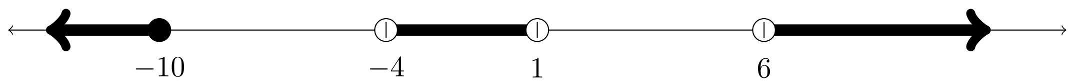
4)

5)

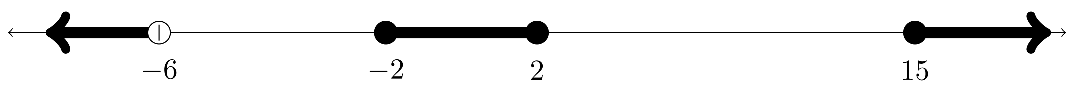


6)

7)

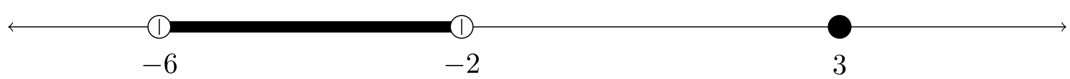


8)



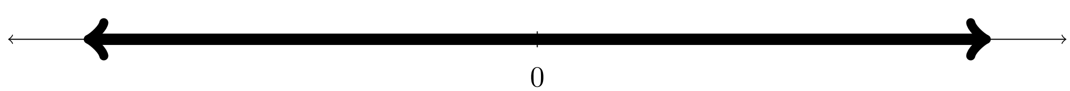
9)

10)

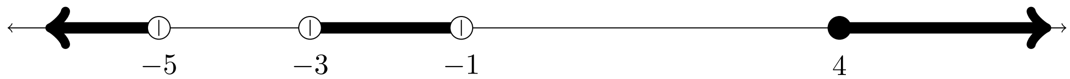


11)

12)



13)



14)

2.2 Solution by Graphing

In previous courses the solution of linear equations is covered, generally by separating the variables and constants on opposite sides of the equation to isolate the variable. In the previous chapter we examined the solution of quadratic equations, in which the variable is isolated using the technique of completing the square. The major difference between these methods of solution is that, in the solving of quadratic equations, we must contend with several different powers of the variable which makes it considerably more difficult to isolate the variable. There are formulas like the quadratic formula available to solve cubic (x^3) and quartic (x^4) equations, however these formulas are somewhat cumbersome and archaic. The primary method for solving eqautions of degree higher than 2 is solution by graphing or by algorithm.

Solution by algorithm is a very interesting process as there are many different algorithms available. Which algorithm is most appropriate often depends on the types of equations being solved and the technology available to solve them. Two major types of algorithms that rely on the graphincal representation of an equation are called “Double False Position” and “The Newton-Raphson Method.” Many commonly available pieces of techonolgy use one of these methods. Since we have graphing calculators available to us, we will focus on solution by graphing.

Whether using a TI (Texas Instruments) or Casio graphing calculator, or a software based graphing ustility such as Graph or Desmos, the solution of these equations focuses on finding the x-intercepts of the graph since this is where the y value is 0. The specific processes for solving equations using each of these different tools will be covered in class.

Example

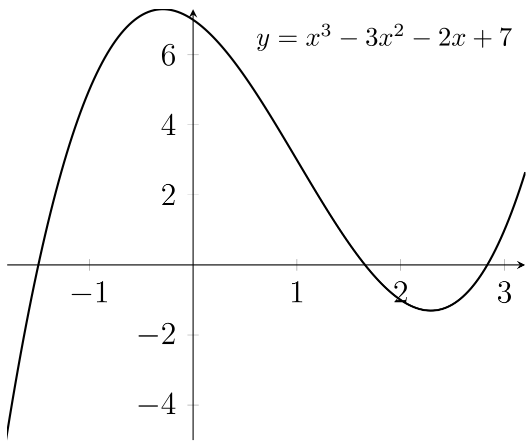
Solve for x.

$x^3 - 3x^2 = 2x - 7$

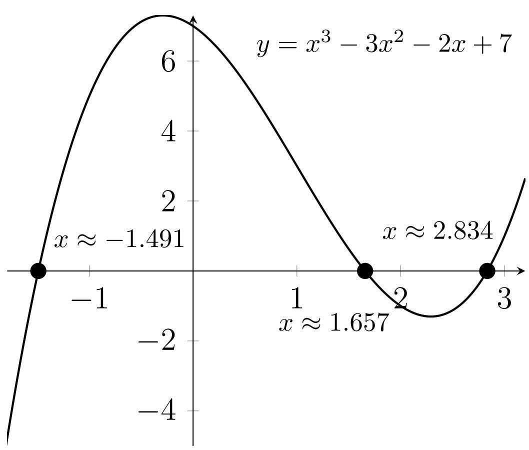
The first step is to move all terms to one side of the equation and set them equal to zero.

$x^3 - 3x^2 - 2x + 7 = 0$

Then we graph the function and look for which x values will make the y value equal to 0.



In this graph, we can see three roots, or x-intercepts, where the graph crosses the x-axis. These are the x-values that make the y-value equal to 0. We can use the available technology to find these x-values.



So, we can see that the solutions to the eqaution $x^3 - 3x^2 - 2x + 7 = 0$ are $x \approx -1.491, 1.657, 2.834$.

We can check these answers by plugging them back into the original equation.

$$\begin{aligned}(-1.491)^3 - 3*(-1.491)^2 - 2*(-1.491) + 7 &= -3.3146 - 3*2.223 + 2.982 + 7 \\ &= -3.3146 - 6.669 + 2.982 + 7 \\ &= -0.0016\end{aligned}$$

The result is not exactly 0 since our answers have been rounded off to the 1000th place. If we wanted greater accuracy, then we should include greater accuracy in the value of our answers.

$$\begin{aligned}(1.657)^3 - 3*(1.657)^2 - 2*(1.657) + 7 &= 4.5495 - 3*2.7456 - 3.314 + 7 \\ &= 4.5495 - 8.2368 - 3.314 + 7 \\ &= -0.0013\end{aligned}$$

and

$$\begin{aligned}(2.834)^3 - 3*(2.834)^2 - 2*(2.834) + 7 &= 22.7614 - 3*8.03155 - 5.668 + 7 \\ &= 22.7614 - 24.09465 - 5.668 + 7 \\ &= -0.00125\end{aligned}$$

Exercises 2.2

Solve for x.

1) $3x^3 - 7x^2 - x + 1 = 0$

2) $24x^4 + 5x^2 - 13x + 3 = 0$

3) $2x^3 - 2x^2 - 28x + 51 = 0$

4) $2x^3 + 5x^2 - 15x + 7 = 0$

5) $x^4 - 4x^3 + x^2 + 6x + 1 = 0$

6) $x^4 + 2x^3 + x^2 - x - 6 = 0$

7) $x^4 - 5x^3 - 3x^2 + 17x - 9 = 0$

8) $2x^3 - 5x - 3 = 0$

9) $x^4 - 4x^3 - 7x^2 - 36x - 18 = 0$

10) $6x^3 - 25x^2 + 21x + 10 = 0$

11) $2x^4 - 5x^3 + x^2 + 4x - 2 = 0$

12) $x^4 + x^3 - 5x^2 + x - 4 = 0$

2.3 Solution of Polynomial Inequalities by Graphing

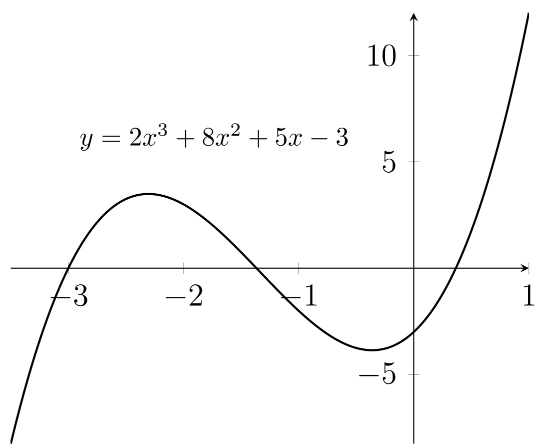
In this section, we will combine the concepts of the previous two sections to solve polynomial inequalities. In Section 2.2, we solved equations by graphing and finding the x-values which made $y = 0$. In solving an inequality, we will be concerned with finding the range of x values that make y either greater than or less than 0, depending on the given problem.

Example

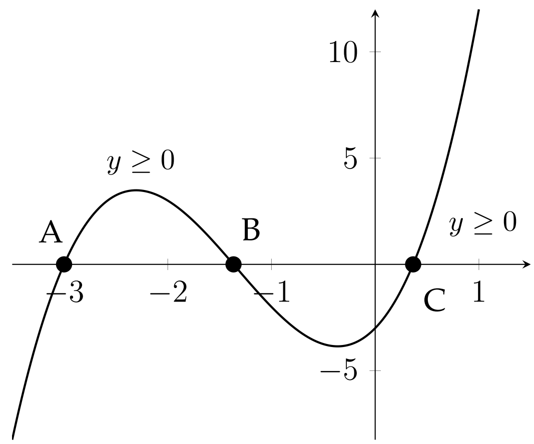
Solve the given inequality.

$2x^3 + 8x^2 + 5x - 3 \geq 0$

First, we graph the function:



Then we identify the intervals of x-values that make the y value greater than or equal to zero, as indicated in the problem.



The indicated roots of the function (A, B and C) are the x-values that make y equal to zero. These points divide the graph between the regions where y is greater than zero and the regions where y is less than zero. The solution to the given inequlaity $2x^3 + 8x^2 + 5x - 3 \geq 0$ are A$\leq x \leq$B OR $x \geq$C.

When we find the values of A, B and C: A$= -3$, B≈ -1.366 and C≈ 0.366, we can complete the solution to the problem.

$2x^3 + 8x^2 + 5x - 3 \geq 0$

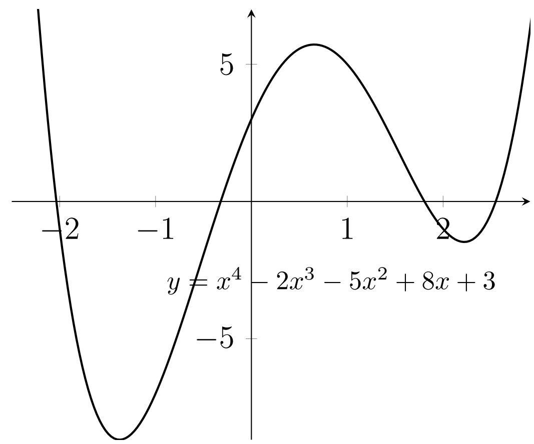
$-3 \leq x \leq -1.366$ OR $x \geq 0.366$

Example

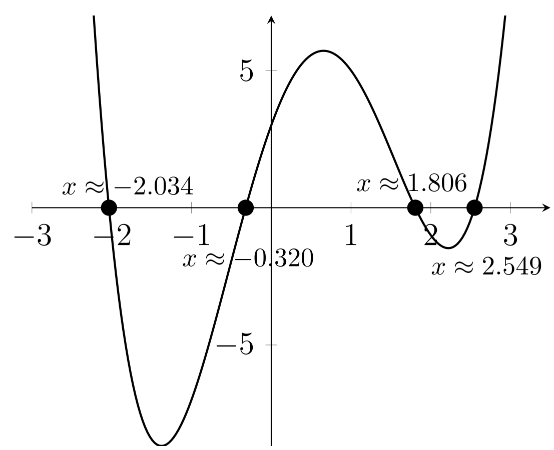
Solve the given inequality.

$x^4 - 2x^3 - 5x^2 + 8x + 3 \leq 0$

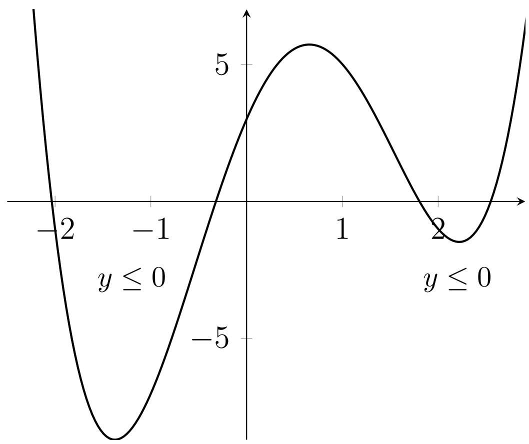
First, we graph the function:



In this problem, we're looking for the intervals of x values that make y less than or equal to zero. First, we identify the roots of the function:



Next, we'll identify the intervals where the y values are less than zero:



So, the solution to the original inequality is:

$x^4 - 2x^3 - 5x^2 + 8x + 3 \leq 0$

$-2.034 \leq x \leq -0.320$ OR $1.806 \leq x \leq 2.549$

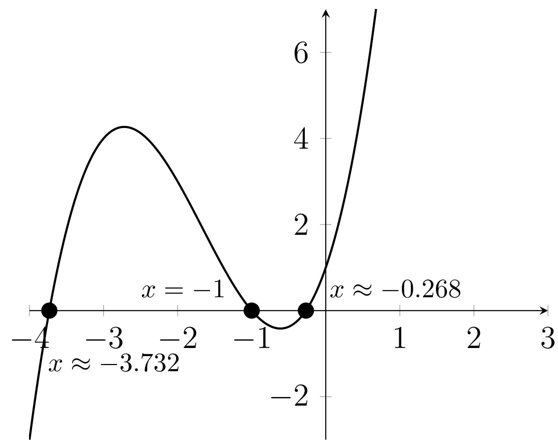
In the next example we'll be looking to identify both the intervals where y is greater than zero, and the intervals where y is less than zero.

Example

Determine the interval(s) for which $x^3 + 5x^2 + 5x + 1 \geq 0$

Determine the interval(s) for which $x^3 + 5x^2 + 5x + 1 < 0$

Once again, we'll start by graphing the function to find the roots:



Now that we've indentified the roots, we can determine where the y values are greater than zero and where they're less than zero.

For $y \geq 0$, we can see that this corresponds to: $-3.732 \leq x \leq -1$ OR $x \geq -0.268$

For $y < 0$, we can see that this corresponds to: $x < -3.732$ OR $-1 < x < -0.268$

Exercises 2.3

1) Determine the interval(s) for which $x^3 - 4x^2 + 2x + 3 \geq 0$

Determine the interval(s) for which $x^3 - 4x^2 + 2x + 3 < 0$

2) Determine the interval(s) for which $4x^3 - 4x^2 - 19x + 10 \geq 0$

Determine the interval(s) for which $4x^3 - 4x^2 - 19x + 10 < 0$

3) Determine the interval(s) for which $x^3 - 2.5x^2 - 7x - 1.5 \geq 0$

Determine the interval(s) for which $x^3 - 2.5x^2 - 7x - 1.5 < 0$

4) Determine the interval(s) for which $x^3 - 3.5x^2 + 0.5x + 5 \geq 0$

Determine the interval(s) for which $x^3 - 3.5x^2 + 0.5x + 5 < 0$

5) Determine the interval(s) for which $6x^4 - 13x^3 + 2x^2 - 4x + 15 \geq 0$

Determine the interval(s) for which $6x^4 - 13x^3 + 2x^2 - 4x + 15 < 0$

6) Determine the interval(s) for which $x^4 - x^3 - x^2 + 3x - 5 \geq 0$

Determine the interval(s) for which $x^4 - x^3 - x^2 + 3x - 5 < 0$

7) Determine the interval(s) for which $3x^4 + 3x^3 - 14x^2 - x + 3 \geq 0$

Determine the interval(s) for which $3x^4 + 3x^3 - 14x^2 - x + 3 < 0$

8) Determine the interval(s) for which $4x^4 - 4x^3 - 7x^2 + 4x + 3 \geq 0$

Determine the interval(s) for which $4x^4 - 4x^3 - 7x^2 + 4x + 3 < 0$

Determine the interval(s) that satisfy each inequality.

9) $x^3 + x^2 - 5x + 3 \leq 0$

10) $x^3 - 7x + 6 > 0$

11) $x^3 - 13x + 12 > 0$

12) $x^4 - 10x^2 + 9 < 0$

13) $6x^4 - 9x^2 - 4x + 12 \geq 0$

14) $x^4 - 5x^3 + 20x - 16 > 0$

15) $x^3 - 2x^2 - 7x + 6 \leq 0$

16) $x^4 - 6x^3 + 2x^2 - 5x + 2 \leq 0$

17) $2x^4 + 3x^3 - 2x^2 - 4x + 2 > 0$

18) $x^5 + 5x^4 - 4x^3 + 3x^2 - 2 \leq 0$

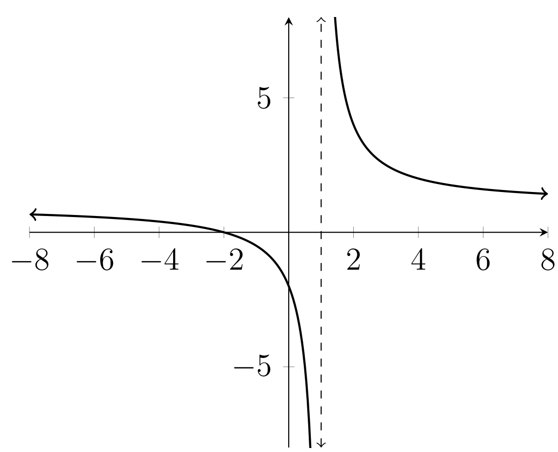
2.4 Solution of Rational Inequalities by Graphing

In the previous section, we saw how to solve polynomial inequalities by graphing. In this section, we will use similar methods to solve rational inequalities. Rational inequalities involve ratios of polynomials or fractions. Because these types of problems involve fractions, the graphs of the functions that we work with will have what are known as *asymptotes*. This word comes from a Greek root having to do with two lines that come very close to each other but never meet.

The vertical asymptotes of a graph will appear at places where the original expression has a zero denominator. This means that the function is not defined at those x values and so, rather than having a y value at that point, the graph has an asymptote.

Example

Below is a graph of the function $y = \dfrac{x+2}{x-1}$



Rather than having a y value at the point where $x = 1$, the dotted line indicates the asymptote where the function is not defined. In the previous section, we were interested in finding the roots of the function because these are the places where $y = 0$, and can be the dividing points between where the y values are greater than zero ($y > 0$) and the where the y values are less than zero ($y < 0$).

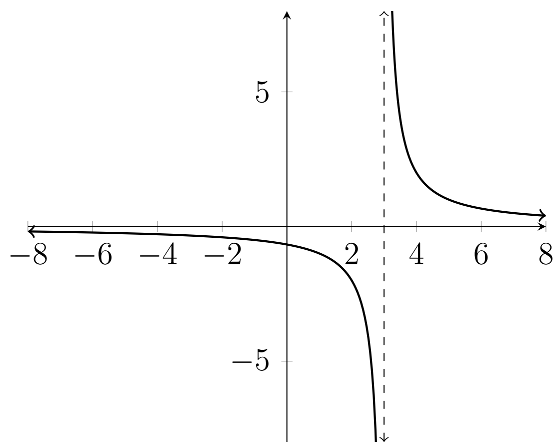
The importance of the asymptotes in analyzing rational functions is that, like the roots, these represent x values that can be the dividing points between where $y > 0$ and where $y < 0$.

Example

Solve the given inequality.

$$\frac{2}{x-3} > 0$$

First we examine the graph:



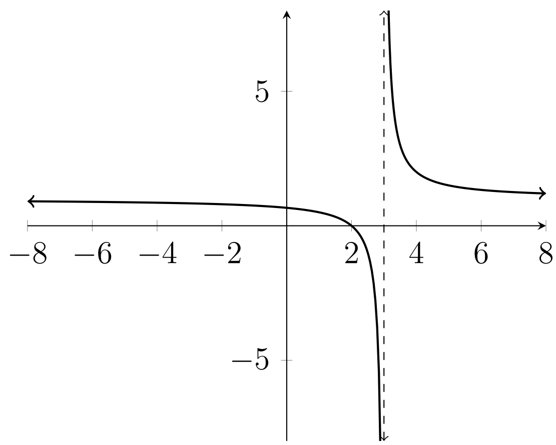
Notice that the asymptote for this graph occurs at the value $x = 3$, because this is the x value that creates a zero denominator. Also notice that the y values switch from being negative to being positive across the asymptote.

There are no roots for this function because there are no x values that make $y = 0$. For a fraction to be zero, the numerator must equal zero. In this example the numerator is 2 and no value of x will make it equal zero. Therefore the only possible dividing point on the graph is $x = 3$, and the solution to the inequality is $x > 3$.

Example

Solve the given inequality.

$$\frac{x-2}{x-3} > 0$$



In this inequality, there is again an asymptote at $x = 3$, but there is also a root at the value $x = 2$, because when $x = 2$. $y = \frac{2-2}{2-3} = \frac{0}{-1} = 0$. So we have two dividing points to consider, $x = 2$ and $x = 3$. We can see from the graph that $y > 0$ for $x < 2$ or $x > 3$, so that is the solution to the given inequality.

Example

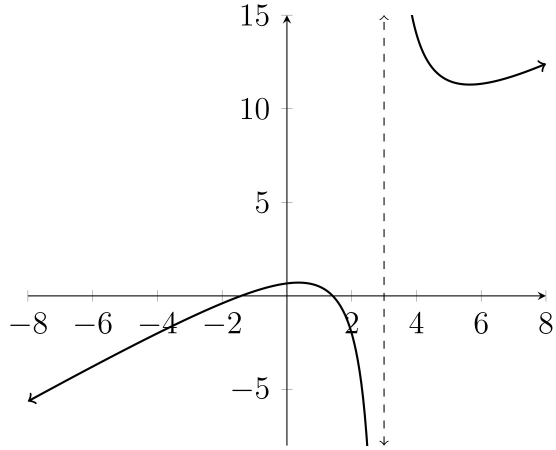
Solve the given inequality.

$$\frac{x^2-2}{x-3} > 0$$

In this problem, we have the same asymptote as the previous two problems: $x = 3$. However, in this inequality, there are two roots, because there are two x values that make the numerator equal zero.

$x^2 - 2 = 0$ means that $x^2 = 2$ and $x = \pm\sqrt{2} \approx \pm 1.414$

We can see these roots on the graph.



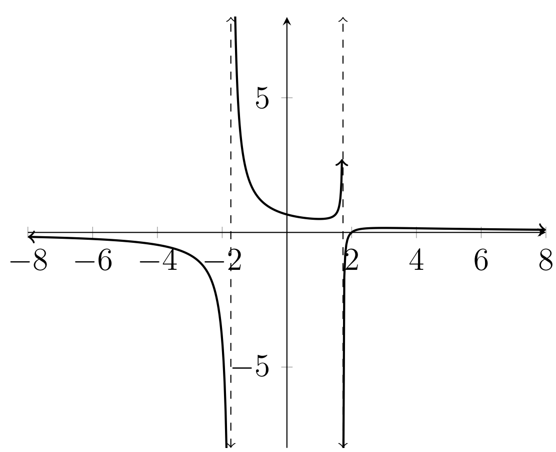
In the graph above, we can see the asymptote at $x = 3$ and the two roots at $x \approx 1.414, -1.414$.

The x values that make $y > 0$ are $-1.414 < x < 1.414$ OR $x > 3$.

Example

Solve the given inequality.

$$\frac{x-2}{x^2-3} > 0$$

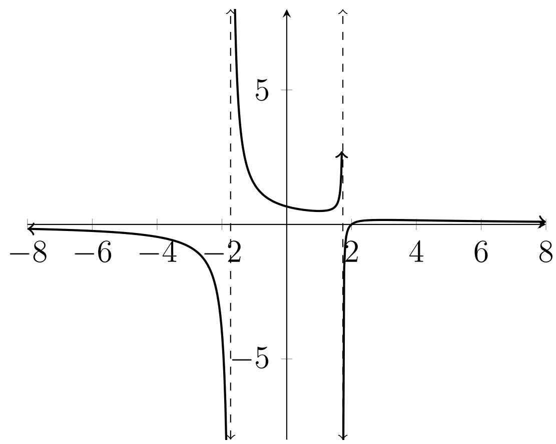


The roots for this function are the x values that make the numerator equal zero:

$x - 2 = 0$, therefore $x = 2$, and we can see this root on the graph.

The asymptotes for the function are the x values that make the denominator equal zero:

$x^2 - 3 = 0$ means that $x^2 = 3$ and $x = \pm\sqrt{3} \approx \pm 1.732$



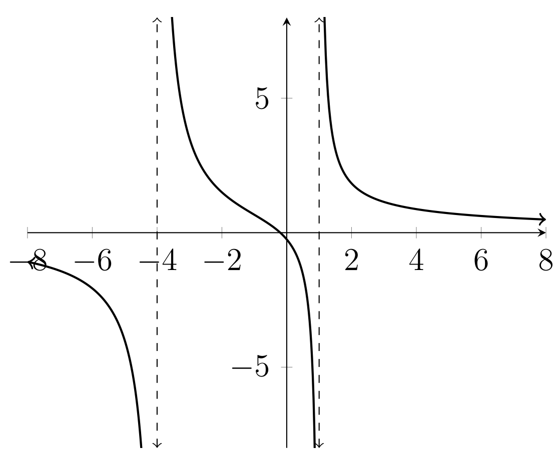
Therefore the solution for the given inequality is:

$-1.732 < x < 1.732$ OR $x > 2$

Example

Solve the given inequality.

$$\frac{5x+1}{x^2+3x-4} < 0$$



Roots

$5x + 1 = 0$

$5x = -1$

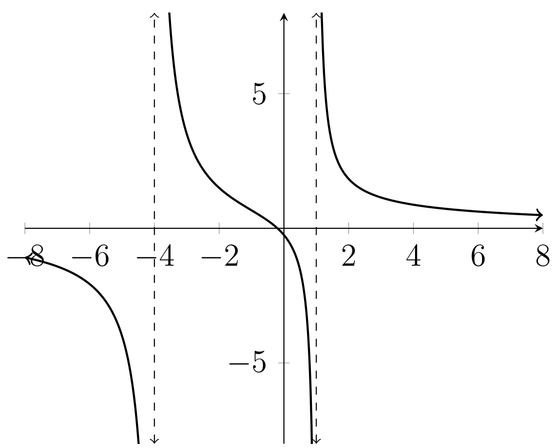
$x = -0.2 = -\frac{1}{5}$

Asymptotes

$x^2 + 3x - 4 = 0$

$(x + 4)(x - 1) = 0$

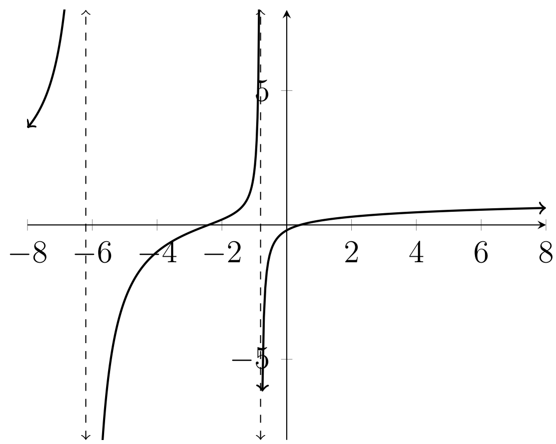
$x = -4, 1$



If we combine the algebraic analysis above with what we see in the graph, then we know that the dividing points important to the solution of this inequality are at $x = -4, -0.2, 1$. The intervals where the y values are less than zero are $x < -4$ OR $-0.2 < x < 1$.

Example

$$\frac{x^2+2x-1}{x^2+7x+5} \leq 0$$



Roots

$x^2+2x-1=0$

$x \approx -2.414, 0.414$

Asymptotes

$x^2+7x+5=0$

$x \approx -6.193, -0.807$

We can see that the dividing points important to the solution of the inequality are $x \approx -6.193, -2.414, -0.807, 0.414$. The intervals where the y values are less than or equal to zero are $-6.193 \leq x \leq -2.414$ OR $-0.807 \leq x \leq 0.414$.

Exercises 2.4

Solve each inequality.

1) $\dfrac{x+4}{x^2-8x+12} > 0$

2) $\dfrac{2x+3}{x^2-2x-35} < 0$

3) $\dfrac{x^2-5x-14}{x^2+3x-10} < 0$

4) $\dfrac{2x^2-x-3}{x^2+10x+16} > 0$

5) $\dfrac{3x+2}{x^2+x-5} < 0$

6) $\dfrac{x^2+2x+5}{x^2-3x-7} > 0$

7) $\dfrac{x^3+9}{x^2+x-1} > 0$

8) $\dfrac{x^3+9}{x^2+x+1} > 0$

Solve each inequality.

9) $\dfrac{x^2-2x-9}{3x+11} > 0$

10) $\dfrac{x^2+4x+3}{2x+1} < 0$

11) $\dfrac{x^2+x-5}{x^2-x-6} > 0$

12) $\dfrac{x^3+2}{x^2-2} > 0$

13) $\dfrac{x^2+2x-7}{x^2+3x-6} < 0$

14) $\dfrac{2x-x^2}{x^2-4x+6} < 0$

15) $\dfrac{x^2-7}{x^2+5x-1} < 0$

16) $\dfrac{x-5}{3x^2-2x-3} > 0$

2.5 Finding Factors from Roots

One method of solving equations involves finding the factors of the polynomial expression in the equation and then setting each factor equal to zero.

$$x^2 + 8x + 15 = 0$$

$$(x + 5)(x + 3) = 0$$

$$x + 5 = 0 \qquad x + 3 = 0$$

$$x = -5 \qquad x = -3$$

In this process, the reasoning is that if $(x + 5)$ times $(x + 3)$ equals zero, then one of those expressions must be equal to zero. In setting them equal to zero, we find the solutions of $x = -5, -3$. Plugging them back into the factored expression we see the following:

$$(-5 + 5)(-5 + 3) = 0 * -2 = 0$$

and

$$(-3 + 5)(-3 + 3) = 2 * 0 = 0$$

This process works in reverse as well. In other words, if we know a root of the function, we can find factors for the expression.

Example

Find a quadratic equation that has roots of -2 and $+3$.

$$x = -2 \qquad x = 3$$

$$x + 2 = 0 \qquad x - 3 = 0$$

$$(x + 2)(x - 3) = 0$$

$$x^2 - x + 6 = 0$$

Roots that are fractions are a little trickier, but really no more difficult:

Example

Find a quadratic equation that has roots of -5 and $\frac{2}{3}$.

$$x = -5 \qquad x = \tfrac{2}{3}$$

$$x + 5 = 0 \qquad 3x = 2$$

$$x + 5 = 0 \qquad 3x - 2 = 0$$

$$(x + 5)(3x - 2) = 0$$

$$3x^2 + 13x - 10 = 0$$

Exercises 2.5

Find a quadratic equation that has the indicated roots.

1) $4, -1$

2) $-2, 7$

3) $\frac{3}{2}, 1$

4) $-\frac{1}{5}, \frac{2}{3}$

5) $\frac{1}{3}, 3$

6) $-4, \frac{2}{5}$

7) $\frac{1}{2}, -\frac{7}{2}$

8) $-1, \frac{3}{5}$

9) $-\frac{2}{3}, -3$

10) $-\frac{2}{3}, -\frac{3}{4}$

11) $-\frac{5}{2}, 3$

12) $-6, -2$

2.6 Polynomial Long Division

Polynomial long division has many similarities to numerical long division, so it is important that we understand how and why numerical long division works the way it does before discussing polynomial long division. First the HOW?

Given the numerical problem $87,462 \div 38$, the first step is to determine the highest place value in the answer.

$$\begin{array}{r} 2 \\ 38\overline{)87,462} \end{array}$$

Often the first step in numerical long division is to say "Does 38 divide into 8?" "No." "Does 38 divide into 87?" "Yes." This tells us that the first digit in the answer will be over the 7 in 87,462, and consequently will be in the thousands place. Once we know that the first digit in the answer will be in the thousands place, the next question is "How many thousands?" We can determine that $2 * 38 = 76$ but $3 * 38 = 114$ (too big), so we know that the first digit in the answer will be 2. Then we subtract, include the 4 and examine what is left over to continue.

$$\begin{array}{l} 2 \\ 38\overline{)87,462} \\ -76 \\ \overline{114} \end{array}$$

Here, we see that $114 \div 38 = 3$, so we know that the next digit in the answer will be 3.

$$\begin{array}{l} 2\,3 \\ 38\overline{)87,462} \\ -76 \\ \overline{114} \\ -114 \\ \overline{0006} \end{array}$$

After including the 6, we can see that 38 does not divide evenly into 6, so we put a zero as the next digit in our answer and proceed:

$$\begin{array}{r} 2\ \ 30 \\ 38\overline{)87,462} \\ -76 \\ \overline{114} \\ -114 \\ \overline{0006} \\ 0 \\ \overline{00062} \end{array}$$

Now that we have included all the digits from our original number 87,462, the last step is to divide 38 into 62. This goes one time with 24 left over.

$$\begin{array}{r} 2\ \ 301 \\ 38\overline{)87,462} \\ -76 \\ \overline{114} \\ -114 \\ \overline{0006} \\ -0 \\ \overline{00062} \\ -38 \\ \overline{00024} \end{array}$$

So, now we have the solution to the original problem $87,462 \div 38 = 2,301$ R24 or $2,301\frac{24}{38}$.

The WHY? of the long division algorithm is somewhat hidden by the HOW? In the first step, we are determining which place value will hold the first digit of our answer. When we determine that 38 does divide into 87, this indicates that the first digit in our answer will be the thousands place. Dividing 38 into 87 tells us how many thousands there will be. Then we subtract:

$$\begin{array}{r} 87,462 \\ -76,000 \\ \hline 11,462 \end{array}$$

Now we need to determine how many times 38 will divide into 11,462. We decided on $300 * 38 = 11,400$, then we subtract to see how much is left over:

$$\begin{array}{r} 11,462 \\ -11,400 \\ \hline 00,062 \end{array}$$

We can see that we won't need any tens in our answer, and that 38 divides into 62 one time with 24 left over, thus the answer is 2 thousands, 3 hundreds, no tens, 1 and a remainder of 24. To check the answer, we multiply $38 * 2301$ and add 24:

$$\begin{array}{l} 2,301 \\ \quad\times 38 \\ \hline 18408 \\ 6903 \\ \hline 87438 \\ \quad +24 \\ \hline 87462 \end{array}$$

Polynomial long division works in much the same way that numerical long division does. Given a problem $A \div B$, the goal is to find a quotient Q and remainder R so that $A = B * Q + R$.

Let's look at this with the example $2x^4 + 7x^3 + 4x^2 - 2x - 1 \div x^2 + 3x + 1$ or:

$$\frac{2x^4 + 7x^3 + 4x^2 - 2x - 1}{x^2 + 3x + 1}$$

So, we are looking to answer the question:

$$\begin{array}{ccccc} A & = & B & * & Q \quad + R \\ 2x^4 + 7x^3 + 4x^2 - 2x - 1 & = & (x^2 + 3x + 1) & * & (?+?+?)+? \end{array}$$

If we want to mulitply $x^2 + 3x + 1$ times something and end up with $2x^4 + 7x^3 + 4x^2 - 2x - 1$, then what we multiply by is going to have to start with $2x^2$, because $x^2 * 2x^2 = 2x^4$.

Now we're working with this:

$$\begin{array}{ccccc} A & = & B & * & Q & + R \\ 2x^4 + 7x^3 + 4x^2 - 2x - 1 & = & (x^2 + 3x + 1) & * & (2x^2 + ? + ?) & + ? \end{array}$$

But the $2x^2$ doesn't just get multiplied by the x^2, it will also get multiplied by the $3x$ and the 1. So now we have:

$$\begin{array}{ccccc} A & = & B & * & Q & + R \\ 2x^4 + 7x^3 + 4x^2 - 2x - 1 & = & (x^2 + 3x + 1) & * & (2x^2 + ? + ?) & + ? \\ & = & 2x^4 + 6x^3 + 2x^2 + ????? \end{array}$$

The issue this raises is that the next multiplication ($? * x^2 + ? * 3x + ? * 1$) needs to add only $1x^3$ to our answer, because we need $7x^3$ and we already have $6x^3$ from the previous multiplication. That means we're going to want to multiply next by $1x$:

$$\begin{array}{ccccc} A & = & B & * & Q & + R \\ 2x^4 + 7x^3 + 4x^2 - 2x - 1 & = & (x^2 + 3x + 1) & * & (2x^2 + 1x + ?) & + ? \\ & = & 2x^4 + 6x^3 + 2x^2 \\ & = & 1x^3 + 3x^2 + x \\ & = & 2x^4 + 7x^3 + 5x^2 + 1x + ??? \end{array}$$

In the next round of mutiplication, we're going to want to bring the $5x^2$ down to $4x^2$, so we'll need to multiply by -1.

$$
\begin{aligned}
A \qquad\qquad &= \qquad B \quad * \quad Q \qquad + R \\
2x^4 + 7x^3 + 4x^2 - 2x - 1 &= (x^2 + 3x + 1) * (2x^2 + 1x - 1) + 0 \\
&= 2x^4 + 6x^3 + 2x^2 \\
&= \qquad 1x^3 + 3x^2 + x \\
&= \qquad\qquad -1x^2 - 3x - 1 \\
&= 2x^4 + 7x^3 + 5x^2 - 2x - 1
\end{aligned}
$$

Now we also know that the remainder is zero, because $x^2 + 3x + 1$ divides evenly into $2x^4 + 7x^3 + 4x^2 - 2x - 1$ and so:

$$2x^4 + 7x^3 + 4x^2 - 2x - 1 = (x^2 + 3x + 1) * (2x^2 + 1x - 1)$$

This method makes the reasoning behind dividing polynomials somewhat more apparent than the long division process, but it is more cumbersome. The way that polynomial long division is usually approached is as follows:

$$
x^2 + 3x + 1 \overline{\big) \quad 2x^4 + 7x^3 + 4x^2 - 2x - 1}
$$

Then, just as we did in the other method, we question "What should we multiply x^2 by to get $2x^4$?" Answer: "$2x^2$" This is the first term in our answer:

$$
\begin{array}{r}
2x^2 \qquad\qquad\quad \\
x^2 + 3x + 1 \overline{\big) \quad 2x^4 + 7x^3 + 4x^2 - 2x - 1}
\end{array}
$$

Then we multiply $2x^2(x^2 + 3x + 1)$ and change all the signs to see what we'll be left with:

$$
\begin{array}{rl}
 & \qquad\qquad\quad 2x^2 \\
x^2 + 3x + 1 \big) & \overline{\quad 2x^4 + 7x^3 + 4x^2 - 2x - 1} \\
 & \underline{-\, 2x^4 - 6x^3 - 2x^2}
\end{array}
$$

This indicates that we'll have the $2x^4$ we'll need in our answer, as well six of the seven x^3's and two of the four x^2's. We'll now need $1x^3$ next:

$$
\begin{array}{r}
2x^2 \\
x^2+3x+1 \overline{\smash{)}\; 2x^4+7x^3+4x^2-2x-1} \\
\underline{-2x^4-6x^3-2x^2} \\
x^3+2x^2-2x
\end{array}
$$

This means we'll need to multiply by $1x$:

$$
\begin{array}{r}
2x^2 \;\; +x \\
x^2+3x+1 \overline{\smash{)}\; 2x^4+7x^3+4x^2-2x-1} \\
\underline{-2x^4-6x^3-2x^2} \\
x^3+2x^2-2x \\
\underline{-x^3-3x^2 \;\; -x} \\
-x^2-3x-1
\end{array}
$$

Here, we still need to pick up a $-1x^2$, which means that our next multiplication will be with -1:

$$
\begin{array}{r}
2x^2 \;\; +x-1 \\
x^2+3x+1 \overline{\smash{)}\; 2x^4+7x^3+4x^2-2x-1} \\
\underline{-2x^4-6x^3-2x^2} \\
x^3+2x^2-2x \\
\underline{-x^3-3x^2 \;\; -x} \\
-x^2-3x-1 \\
\underline{x^2+3x+1} \\
0
\end{array}
$$

Because x^2+3x+1 divides evenly into $2x^4+7x^3+4x^2-2x-1$ we have a zero remainder. In the next example there will be a non-zero remainder:

Example

Divide:

$$\frac{3x^4-8x^3+19x^2-15x+10}{x^2-x+4}$$

First, we set up the problem:

$$x^2 - x + 4 \overline{\big)\; 3x^4 - 8x^3 + 19x^2 - 15x + 10}$$

Then, we question: "What do we need to multiply x^2 by to get $3x^4$?" Answer: "$3x^2$"

$$\begin{array}{r}
3x^2 \\
x^2 - x + 4 \overline{\big)\; 3x^4 - 8x^3 + 19x^2 - 15x + 10}
\end{array}$$

Then, we multiply, change signs (subtract) and combine like terms:

$$\begin{array}{r}
3x^2 \qquad\qquad \\
x^2 - x + 4 \overline{\big)\; 3x^4 - 8x^3 + 19x^2 - 15x + 10} \\
\underline{-3x^4 + 3x^3 - 12x^2} \qquad\qquad \\
-5x^3 + 7x^2 - 15x \qquad
\end{array}$$

Now we'll need to multiply by $-5x$, change signs and combine like terms:

$$\begin{array}{r}
3x^2 - 5x \qquad \\
x^2 - x + 4 \overline{\big)\; 3x^4 - 8x^3 + 19x^2 - 15x + 10} \\
\underline{-3x^4 + 3x^3 - 12x^2} \qquad\qquad \\
-5x^3 + 7x^2 - 15x \qquad \\
\underline{5x^3 - 5x^2 + 20x} \qquad \\
2x^2 + 5x + 10
\end{array}$$

We need $2x^2$ so we'll need to multiply by 2, change signs and combine like terms:

$$\begin{array}{r}
3x^2 - 5x + 2 \\
x^2 - x + 4 \overline{\big)\; 3x^4 - 8x^3 + 19x^2 - 15x + 10} \\
\underline{-3x^4 + 3x^3 - 12x^2} \qquad\qquad \\
-5x^3 + 7x^2 - 15x \qquad \\
\underline{5x^3 - 5x^2 + 20x} \qquad \\
2x^2 + 5x + 10 \\
\underline{-2x^2 + 2x - 8} \\
7x + 2
\end{array}$$

Because there is no positive power of x that we can multiply x^2 by to get $7x$, then this is our remainder: $7x + 2$.

So:

$$\begin{array}{ccccccc} A & = & B & * & Q & + & R \\ 3x^4 - 8x^3 + 19x^2 - 15x + 10 & = & (x^2 - x + 4) & * & (3x^2 - 5x + 2) & + & (7x + 2) \end{array}$$

Exercises 2.6

Find the quotient in each problem.

1) $\dfrac{y^3 - 4y^2 + 6y - 4}{y - 2}$

2) $\dfrac{x^3 - 5x^2 + x + 15}{x - 3}$

3) $\dfrac{x^3 - 4x^2 - 3x - 10}{x^2 + x + 2}$

4) $\dfrac{2x^3 - 3x^2 + 7x - 3}{x^2 - x + 3}$

5) $\dfrac{x^4 + 2x^3 - x^2 + x + 6}{x + 2}$

6) $\dfrac{x^4 + x^3 + 5x^2 + 3x + 6}{x^2 + x - 1}$

7) $\dfrac{2z^3 + 5z + 8}{z + 1}$

8) $\dfrac{x^5 + 3x + 2}{x^3 + 2x + 1}$

9) $\dfrac{x^4 + 2x^3 + 4x^2 + 3x + 2}{x^2 + x + 2}$

10) $\dfrac{2x^4 + 3x^3 + 3x^2 - 5x - 3}{2x^2 - x - 1}$

11) $\dfrac{2y^5 - 3y^4 - y^2 + y + 4}{y^2 + 1}$

12) $\dfrac{3x^5 - 4x^3 + 3x^2 + 12x - 10}{x^2 + 2x - 1}$

13) $\dfrac{5x^4 - 3x^2 + 2}{x^2 - 3x + 5}$

14) $\dfrac{3y^3 - 4y^2 - 3}{y^2 + 5y + 2}$

2.7 Synthetic Division

The process for polynomial long division (like the process for numerical long division) has been separated somewhat from its logical underpinnings for a more efficient method to arrive at an answer. For particular types of polynomial long division, we can even take this abstraction one step further. Synthetic Division is a handy shortcut for polynomial long division problems in which we are dividing by a *linear* polynomial. This means that the highest power of x we are dividing by needs to be x^1. This limits the usefulness of Synthetic Division, but it will serve us well for certain purposes. Let's examine where the coefficients in our answer come from when we divide by a linear polynomial:

$$\begin{array}{r|l} & \textcircled{2}x^3 \\ \hline x-5 & \textcircled{2}x^4 - 6x^3 - 23x^2 + 16x - 5 \\ & -2x^4 + 10x^3 \\ \cline{2-2} & 4x^3 - 23x^2 \end{array}$$

Notice that the first coefficient in the answer is the same as the first coefficient in the polynomial we're dividing into. This is because we're dividing by a polynomial in the form $1x^1 - a$. This also makes the power of the first term in the answer one less than the power of the polynomial we are dividing into. Let's see where the subsequent coefficients in the answer come from:

$$\begin{array}{r|l} & 2x^3 \textcircled{+4}x^2 \\ \hline x-5 & 2x^4 - 6x^3 - 23x^2 + 16x - 5 \\ & -2x^4 + 10x^3 \\ \cline{2-2} & \textcircled{4}x^3 - 23x^2 \\ & -4x^3 + 20x^2 \\ \cline{2-2} & -3x^2 + 16x \end{array}$$

The next coefficient in the answer (4) comes from the combination of the -6 and the $+10$. The $+10$ came from multiplying the 2 in the answer by the 5 in the divisor $x - 5$. The next coefficient in the answer will be -3, which comes from multiplying the 4 (in the answer) by the 5 (in the divisor) and combining it with the -23 in the polynomial we're dividing into:

$$
\begin{array}{rrrrrr}
 & & 2x^3 & +4x^2 & -3x & \\
\cline{2-6}
x-5 \big) & 2x^4 & -6x^3 & -23x^2 & +16x & -5 \\
 & -2x^4 & +10x^3 & & & \\
\cline{2-3}
 & & 4x^3 & -23x^2 & & \\
 & & -4x^3 & +20x^2 & & \\
\cline{3-4}
 & & & -3x^2 & +16x & \\
 & & & 3x^2 & -15x & \\
\cline{4-5}
\end{array}
$$

The last part of our answer will come from multiplying the -3 in the answer times the 5 in the divisor (making -15) and combining this with the $+16$ in the polynomial we're dividing into:

$$
\begin{array}{rrrrrr}
 & & 2x^3 & +4x^2 & -3x & +1 \\
\cline{2-6}
x-5 \big) & 2x^4 & -6x^3 & -23x^2 & +16x & -5 \\
 & -2x^4 & +10x^3 & & & \\
\cline{2-3}
 & & 4x^3 & -23x^2 & & \\
 & & -4x^3 & +20x^2 & & \\
\cline{3-4}
 & & & -3x^2 & +16x & \\
 & & & 3x^2 & -15x & \\
\cline{4-5}
 & & & & 1x & -5 \\
 & & & & -x & +5 \\
\cline{5-6}
 & & & & & 0
\end{array}
$$

The process of Synthetic Division uses these relationships as a shortcut to finding the answer. The set-up for a Synthetic Division problem is shown below:

$$
\begin{array}{r|rrrrr}
5 & 2 & -6 & -23 & 16 & -5 \\
 & & & & & \\
\hline
 & & & & &
\end{array}
$$

This set-up allows us to complete the division problem $\dfrac{2x^4 - 6x^3 - 23x^2 + 16x - 5}{x - 5}$.

$$\begin{array}{r|rrrrr} 5 & 2 & -6 & -23 & 16 & -5 \\ & & & & & \\ \hline & & & & & \end{array}$$

The first coefficient in the answer is the same as the first coefficient in the polynomial we're dividing into:

$$\begin{array}{r|rrrrr} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & & & & \\ \hline & 2 & & & & \end{array}$$

To get the next coefficient, we multiply the 2 by the 5 to get $+10$ and flll this in under the -6:

$$\begin{array}{r|rrrrr} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & & & \\ \hline & 2 & & & & \end{array}$$

Then, $-6 + 10 = +4$, which is the next coefficient in the answer:

$$\begin{array}{r|rrrrr} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & & & \\ \hline & 2 & 4 & & & \end{array}$$

Then, we continue this process, multiplying the 4 by the 5 to get 20 and combining this with the -23: $\quad -23 + 20 = -3$:

$$\begin{array}{r|rrrrr} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & +20 & & \\ \hline & 2 & 4 & -3 & & \end{array}$$

Next, multiply the -3 by the 5 and combine the resulting -15 with the 16:

$$\begin{array}{r|rrrrr} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & +20 & -15 & \\ \hline & 2 & 4 & -3 & 1 & \end{array}$$

In the last step, multiply the 1 times the 5 and combine the result with the -5 in the problem to get zero:

$$\begin{array}{r|rrrr|r} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & +20 & -15 & 5 \\ \hline & 2 & 4 & -3 & 1 & 0 \end{array}$$

This last coefficient represents the remainder - in this case 0. The other numerals in the answer represent the coefficients for the powers of x in the answer. On the far right is the remainder, then the constant (x^0) term, then the linear (x^1) term and so on:

$$\begin{array}{r|rrrr|r} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & +20 & -15 & 5 \\ \hline & 2x^3 & 4x^2 & -3x^1 & 1x^0 & 0 \end{array}$$

$$\frac{2x^4 - 6x^3 - 23x^2 + 16x - 5}{x - 5} = 2x^3 + 4x^2 - 3x + 1$$

Let's look at another example:

Example

Use Synthetic Division to divide:

$$\frac{3x^3 + 5x^2 - 9x + 9}{x + 3}$$

Since Synthetic Division is set up to divide by $x - a$, if we're dividing by $x + 3$ we'll need to use a -3 in the Synthetic Division:

$$\begin{array}{r|rrrr} -3 & 3 & 5 & -9 & 9 \\ & \downarrow & & & \\ \hline & 3 & & & \end{array}$$

Then, $3 * -3 = -9$:

$$\begin{array}{r|rrrr} -3 & 3 & 5 & -9 & 9 \\ & \downarrow & -9 & & \\ \hline & 3 & -4 & & \end{array}$$

Next, $-4 * -3 = +12$:

$$\begin{array}{r|rrrr} -3 & 3 & 5 & -9 & 9 \\ & \downarrow & -9 & 12 & \\ \hline & 3 & -4 & 3 & \end{array}$$

And this example also has a zero remainder:

$$\begin{array}{r|rrr|r} -3 & 3 & 5 & -9 & 9 \\ & \downarrow & -9 & 12 & -9 \\ \hline & 3 & -4 & 3 & 0 \end{array}$$

The answer here is $3x^2 - 4x + 3$:

$$\frac{3x^3 + 5x^2 - 9x + 9}{x + 3} = 3x^2 - 4x + 3$$

and

$$3x^3 + 5x^2 - 9x + 9 = (x + 3)(3x^2 - 4x + 3)$$

Let's look at an example that is a little bit different.

Example

Use Synthetic Division to divide:

$$\frac{6x^4 + x^3 + 9x^2 + x - 2}{2x + 1}$$

Synthetic Division is set up to handle problems in which we are dividing by $1x - a$. Clearly, this is not the case in this example, however we can work around this. Another way of looking at setting up the synthetic division is that we use the number that is the solution to $x - a = 0$. When we divided by $x - 5$, we used $+5$. When we were dividing by $x + 3$, we used -3. So, if we're going to divide by $2x + 1$, we'll use $-\frac{1}{2}$ in the Synthetic Division:

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 6 & 1 & 9 & 1 & -2 \\ & \downarrow & & & & \\ \hline & 6 & & & & \end{array}$$

Then, we proceed as usual:

$$\begin{array}{r|rrrr|r} -\frac{1}{2} & 6 & 1 & 9 & 1 & -2 \\ & \downarrow & -3 & 1 & -5 & 2 \\ \hline & 6 & -2 & 10 & -4 & 0 \end{array}$$

Notice that, again, we have a zero remainder. Also, notice that each coefficient in our answer has a common factor of 2, which was the coefficient of the x in $2x+1$, which we orginally were going to divide by. What we've done here is not division by $2x+1$, but division by $x+\frac{1}{2}$.

So, in the end, our work tells us that:

$$\frac{6x^4+x^3+9x^2+x-2}{x+\frac{1}{2}} = 6x^3-2x^2+10x-4$$

and

$$6x^4+x^3+9x^2+x-2 = \left(x+\frac{1}{2}\right)(6x^3-2x^2+10x-4)$$

Notice that if we factor out the common factor of 2 from our answer, we can multiply it back into the $x+\frac{1}{2}$ and get an answer for our original problem:

$$6x^4+x^3+9x^2+x-2 = \left(x+\frac{1}{2}\right)2(3x^3-x^2+5x-2)$$

$$= (2x+1)(3x^3-x^2+5x-2)$$

This means that:

$$\frac{6x^4+x^3+9x^2+x-2}{2x+1} = 3x^3-x^2+5x-2$$

Another thing to understand about Synthetic Division is that if there is a missing power of x, then you should include a zero as the coefficient of that power.

Example

Use Synthetic Division to divide:

$$\frac{x^3 + 4x - 6}{x - 2}$$

Since there is no x^2 term in the polynomial we're divding into, we'll enter a zero as the coefficient for that term:

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 4 & -6 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

And then proceed as usual:

$$\begin{array}{r|rrr|r} 2 & 1 & 0 & 4 & -6 \\ & \downarrow & 2 & 4 & 16 \\ \hline & 1 & 2 & 8 & 10 \end{array}$$

So the answer for this problem is $x^2 + 2x + 8 \qquad R: 10$.

Exercises 2.7

Use synthetic division to find the quotient in each problem.

1) $\dfrac{x^3 - 8x^2 + 5x + 50}{x - 5}$

2) $\dfrac{x^3 + 5x^2 - x - 14}{x + 2}$

3) $\dfrac{x^3 + 12x^2 + 34x - 7}{x + 7}$

4) $\dfrac{x^3 - 10x^2 + 23x - 6}{x - 3}$

5) $\dfrac{x^4 - 15x^2 + 10x + 24}{x + 4}$

6) $\dfrac{x^4 - 3x^3 + 4x^2 - 36}{x - 3}$

7) $\dfrac{x^4 - 2x^3 - x + 10}{x - 2}$

8) $\dfrac{x^4 - 16x^2 - 5x - 24}{x + 4}$

9) $\dfrac{2x^4 - x^3 + 2x - 1}{2x - 1}$

10) $\dfrac{3x^4 + x^3 - 3x + 1}{3x + 1}$

11) $\dfrac{3x^4 - 8x^3 + 9x^2 - 2x - 2}{3x + 1}$

12) $\dfrac{6x^4 - 7x^3 + 5x^2 - 17x + 10}{3x - 2}$

13) $\dfrac{2x^3 + 7x^2 + 6x - 5}{2x - 1}$

14) $\dfrac{3x^4 - x^3 - 21x^2 - 11x + 6}{3x - 1}$

2.8 Roots and Factorization of Polynomials

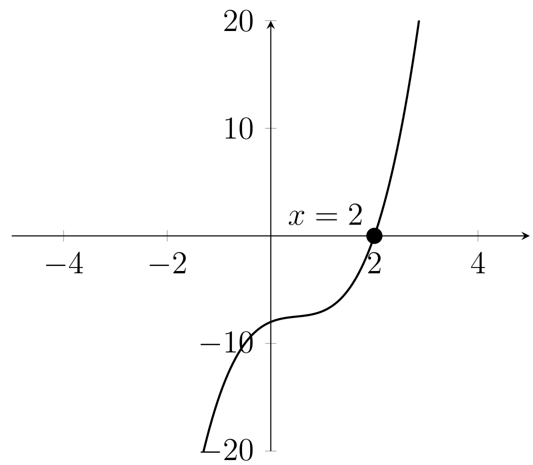
In this section we will use some of the skills we have seen in previous sections in order to find all the roots of a polynomial function (both real and complex) and also factor the polynomial as the product of prime factors with integer coefficients.

Example

Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.

$2x^3 - 3x^2 + 2x - 8 = 0$

First we'll graph the polynomial to see if we can find any real roots from the graph:



We can see that there is a root at $x = 2$. This means that the polynomial will have a factor of $(x - 2)$. We can use Synthetic Division to find any other factors. Because $x = 2$ is a root, we should get a zero remainder:

2	2	−3	2	−8
	↓	4	2	8
	2	1	4	0

So, now we know that $2x^3 - 3x^2 + 2x - 8 = (x-2)(2x^2 + x + 4)$. To finish the problem, we can set each factor equal to zero and find the roots:

$$2x^3 - 3^2 + 2x - 8 = 0$$

$$(x-2)(2x^2 + x + 4) = 0$$

$$x - 2 = 0 \qquad 2x^2 + x + 4 = 0$$

$$x = 2 \qquad x \approx -0.25 \pm 1.392i$$

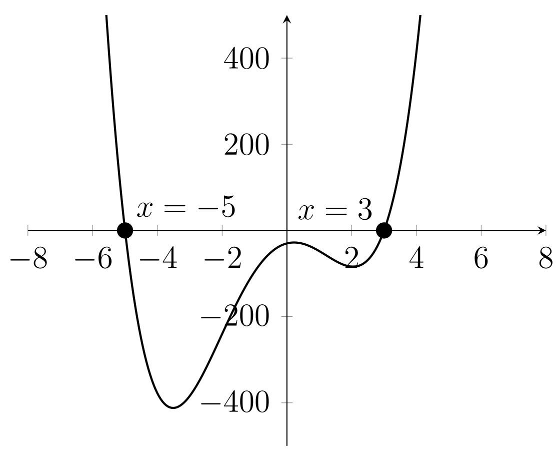
Let's look at an example that has more than one real root:

Example

Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.

$3x^4 + 5x^3 - 45x^2 + 19x - 30 = 0$

First we'll graph the polynomial to see if we can find any real roots from the graph:



We can see roots at $x = -5, 3$, which means that $(x+5)$ and $(x-3)$ are both factors of this polynomial. We'll need to divide by both of these factors to break down the polynomial. First, we divide by $(x-3)$:

$$\begin{array}{r|rrrr|r} 3 & 3 & 5 & -45 & 19 & -30 \\ & \downarrow & 9 & 42 & -9 & 30 \\ \hline & 3 & 14 & -3 & 10 & 0 \end{array}$$

And then by $(x+5)$:

$$\begin{array}{r|rr r|r|r} 3 & 3 & 5 & -45 & 19 & -30 \\ & \downarrow & 9 & 42 & -9 & 30 \\ \hline -5 & 3 & 14 & -3 & 10 & 0 \\ & \downarrow & -15 & 5 & -10 & \\ \hline & 3 & -1 & 2 & 0 & \end{array}$$

Now we know that $3x^4 + 5x^3 - 45x^2 + 19x - 30 = (x+5)(x-3)(3x^2 - x + 2)$ and so, to finish the problem:

$$3x^4 + 5x^3 - 45x^2 + 19x - 30 = 0$$

$$(x+5)(x-3)(3x^2 - x + 2) = 0$$

$$x + 5 = 0 \qquad x - 3 = 0 \qquad 3x^2 - x + 2 = 0$$

$$x = -5 \qquad x = 3 \qquad x \approx \tfrac{1}{6} \pm 0.799i$$

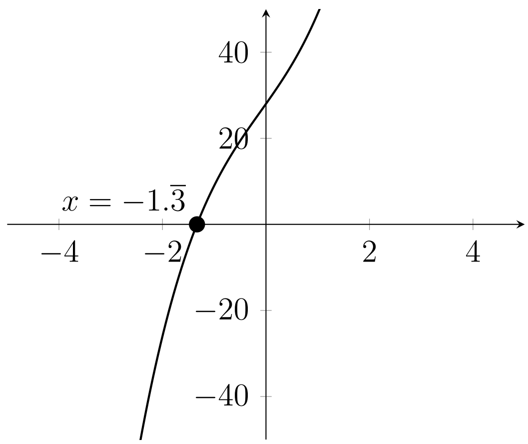
Next, let's look at an example where there is a root that is not a whole number:

Example

Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.

$3x^3 + x^2 + 17x + 28 = 0$

First we'll graph the polynomial to see if we can find any real roots from the graph:



We can see in the graph that this polynomial has a root at $x = -\frac{4}{3}$. That means that the polynomial must have a factor of $3x + 4$. We can use Synthetic Division to find the other factor for this polynomial. Because we know that $x = -\frac{4}{3}$ is a root, we should get a zero remainder:

$-\frac{4}{3}$	3	1	17	28
	↓	−4	4	−28
	3	−3	21	0

Notice that, because the root we used was a fraction, there is a common factor of 3 in the answer to our Synthetic Division. We should factor this out to obtain the answer:

$(x + \frac{4}{3})(3x^2 - 3x + 21) = (3x + 4)(x^2 - x + 7)$

So, this means that:

$$3x^3 + x^2 + 17x + 28 = 0$$

$$(3x + 4)(x^2 - x + 7) = 0$$

$$3x + 4 = 0 \qquad x^2 - x + 7 = 0$$

$$x = -\frac{4}{3} \qquad x \approx 0.5 \pm 2.598i$$

Exercises 2.8

Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.

Set #1

1) $x^4 - 3x^3 + 5x^2 - x - 10 = 0$

2) $3x^3 - 5x^2 + 2x - 8 = 0$

3) $2x^4 - 5x^3 + x^2 + 4x - 4 = 0$

4) $x^4 + x^3 - 3x^2 - 17x - 30 = 0$

5) $x^4 - 9x^3 + 21x^2 + 21x - 130 = 0$

6) $x^4 - 7x^3 + 14x^2 - 38x - 60 = 0$

7) $x^5 - 9x^4 + 31x^3 - 49x^2 + 36x - 10 = 0$

8) $x^4 + 4x^3 + 2x^2 + 12x + 45 = 0$

9) $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$

10) $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$

11) $x^5 - 3x^4 + 12x^3 - 28x^2 + 27x - 9 = 0$

12) $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0$

Set #2

13) $15x^3 - 7x^2 + 13x + 3 = 0$

14) $x^4 - 5x^3 + 3x^2 - 11x - 20 = 0$

15) $6x^3 + 13x^2 + 12x + 4 = 0$

16) $6x^3 - 5x^2 + 5x - 2 = 0$

17) $4x^4 + 20x^3 + 29x^2 + 10x - 15 = 0$

18) $3x^4 - 4x^3 + 10x^2 + 12x - 5 = 0$

19) $2x^4 - 3x^3 - 6x^2 - 8x - 3 = 0$

20) $12x^4 - 53x^3 - 31x^2 - 19x - 5 = 0$

21) $12x^4 + 4x^3 + x^2 - 3x - 2 = 0$

22) $3x^4 + 13x^3 - 26x - 40 = 0$

Chapter 3

Exponents and Logarithms

In this chapter, we will examine concepts that are related to exponential, logarithmic and logistic relationships. In the first section, we will look at how to approach these problems from a graphical perspective. In the subsequent sections, we will examine the methods necessary to work with these problems algebraically.

3.1 Exponential and Logistic Applications

There are a variety of different types of mathematical relationships. The simplest mathematical relationship is the additive relationship. This is a situation in which the value of one quantity is always a certain amount more (or less) than another quantity. A good example of an additive relationship is an age relationship. In an age relationship, the age of the older person is always the same amount more than the age of the younger person. If the older person is five years older, then the age of the older person (y) will always be equal to the age of the younger person (x) plus five: $y = x + 5$.

Another type of additive relationship is seen where two quantities add up to a constant value. Let's say there is a board whose length is 20 inches. If we cut it into two pieces, with one piece being 6 inches, then the other piece will be 14 inches. If one piece is 9 inches, then the other will be 11 inches. If one piece is x inches, then the other piece (y) will be $20 - x$: $y = 20 - x$ or $x + y = 20$.

The next type of mathematical relationshp is a multiplicative relationship. This represents a situation in which one quantity is always a multiple of the other quantity. This is commonly seen in proportional relationships. If a recipe for a cake calls for 2 cups of flour, then, if we want to make 3 cakes, we'll need 6 cups of flour. The amount of flour (y) is always two times the number of cakes we want to make (x): $y = 2x$.

If a recipe for a batch of cookies (with 20 cookies per batch) calls for 1.5 cups of sugar, then three batches would require 4.5 cups of sugar. The amount of sugar required (y) is always the number of batches (x) times 1.5: $y = 1.5x$. If we wanted to represent this relationship based on the number of cookies instead of the number of batches, we would need to adjust the formula. Given that there are 20 cookies per batch, we could adjust our formula so that we first calculate the number of batches from the number of cookies and then multiply by 1.5. If the number of cookies is x and the amount of sugar is y, then $y = 1.5 * \frac{x}{20}$ or $y = \frac{3}{40}x$.

The next type of mathematical relationship is the polynomial relationship. In this type of relationship, one quantity is related to a power of another quantity. A good example of this type of relationship involves gravity. As Galileo discovered in the 16th century, the distance that an object falls after it is dropped is not proportional to the time that it has been falling. Rather, it is proportional to the square of the time. The table below shows this type of relationship.

t	d
1	16
2	64
3	144
4	256

After one second, it looks like the distance will always be sixteen times the time the object has been falling. However, after two seconds, we can see that this relationship no longer is true. That's because this relationship is a polynomial relationship in which the distance an object has fallen (d) is proportional to the *square* of the time it has been falling (t): $d = 16t^2$.

Exponential Relationships

The next type of relationship is the focus of this chapter - the *exponential* relationship. In this situation, the rate of change of a quantity is proportional to the size of that quantity. This relationship can be explored in more depth in an integral calculus course, but we will discuss the basics here.

In a linear or proportional relationship, the slope, or rate of change, is constant. For example, in the equation $y = 3x + 1$, the slope is always three, no matter what the values of x and y are. In an exponential relationship, the rate of change (also called "y prime" or y') is proportional to the value of y. In this case, we say that $y' = k * y$.

This is what is known as a differential equation. This is an equation in which the variable and its rate of change are related. Through the processes of differential and integral calculus, we can solve the equation above $y' = k * y$ as:

$$y = Ae^{kt}$$

In the equation above, A is the value of y at time $t = 0$, k is a constant that determines how fast the quantity y increases or decreases and t plays the role of the independent variable (as x often does) and represents the time that has passed. If k is positive, then the quantity y is growing because its rate of change is positive. If k is negative, then the quantity y is decreasing because the rate of change is negative.

The quantity represented by e in the above equation is a mathematical constant (like π) that is often used to represent exponential relationships. The best way to understand the value of e and what it represents is directly related to fundamental questions from differential and integral calculus.

Differential Calculus is concerned primarily with the question of slopes. We discussed earlier that a linear relationship has a constant slope. Polynomial and exponential relationships have slopes that depend on the value of x and/or y. This is what makes them curves rather than lines. If we consider the slopes of some different exponential relationships, we can see one aspect of where the value for e comes from.

Consider the graphs of the following relationships:

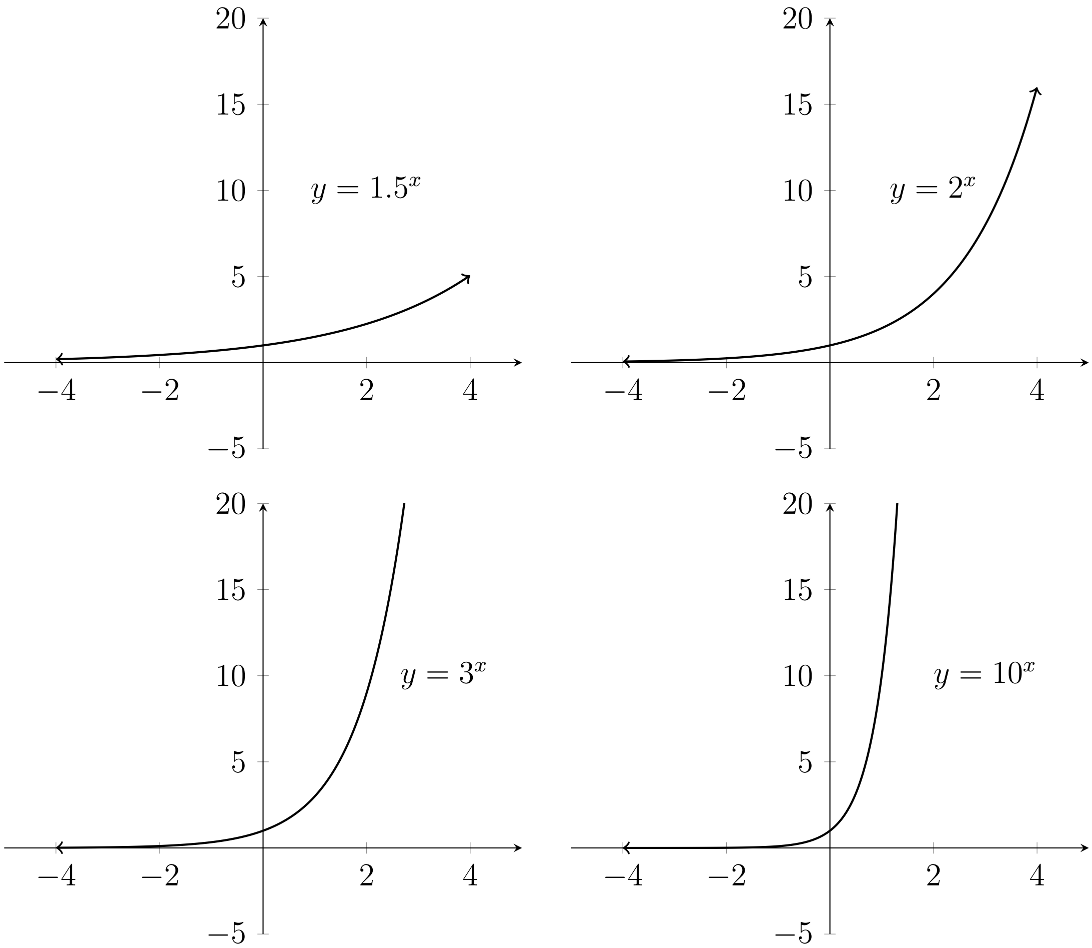
$$y = 1.5^x$$

$$y = 2^x$$

$$y = 3^x$$

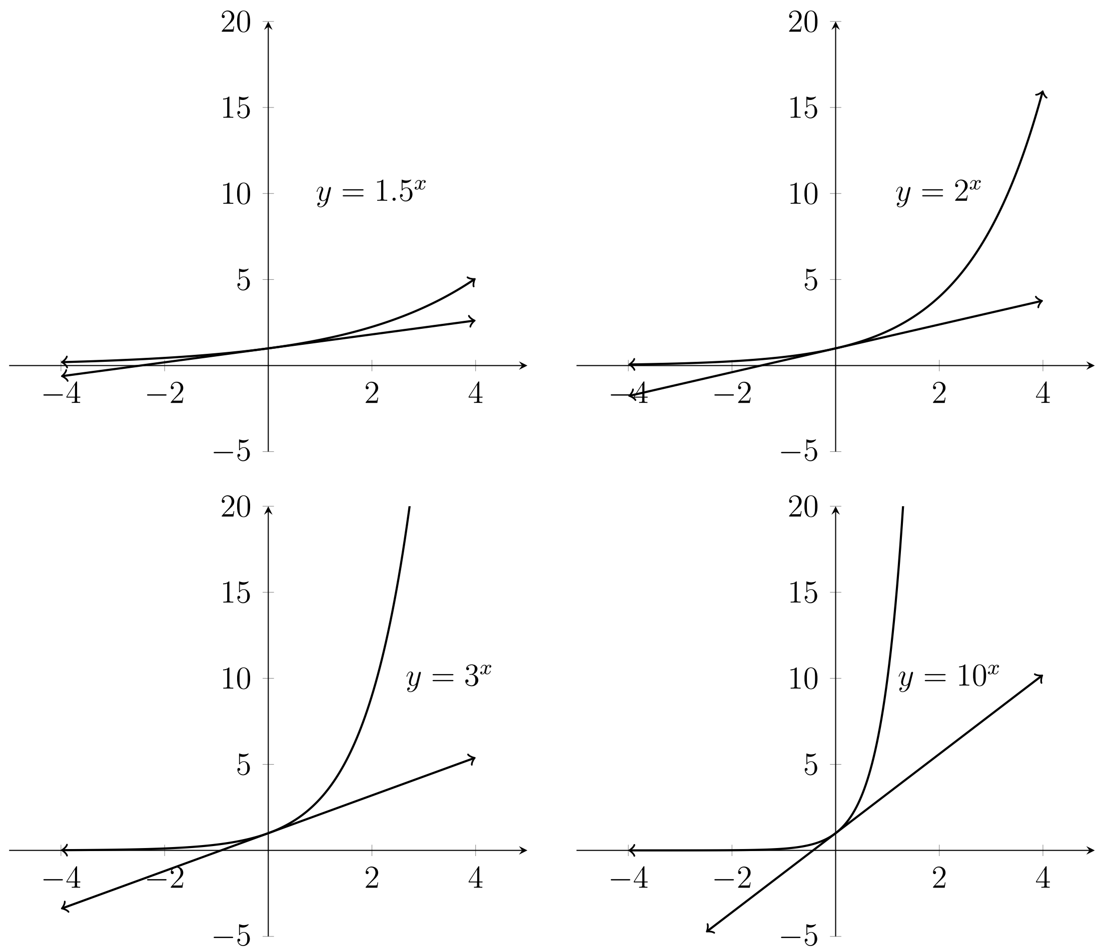
$$y = 10^x$$

Let's look at the graphs for these functions:



We can see that these graphs demonstrate slightly different behavior and different x and y values. One thing that they all have in common is that they all pass through the point $(0,1)$ on the graph. This is because $1.5^0 = 1$, $2^0 = 1$, $3^0 = 1$ and $10^0 = 1$. Therefore the point where x is 0 and y is 1 is on all four of the graphs.

Although all four of the graphs pass through the point $(0,1)$, they each do this in a different way. Let's look at the slope of a line *tangent* to each curve at the point $(0,1)$. This is the straight line that touches the curve at the point $(0,1)$, but nowhere else:

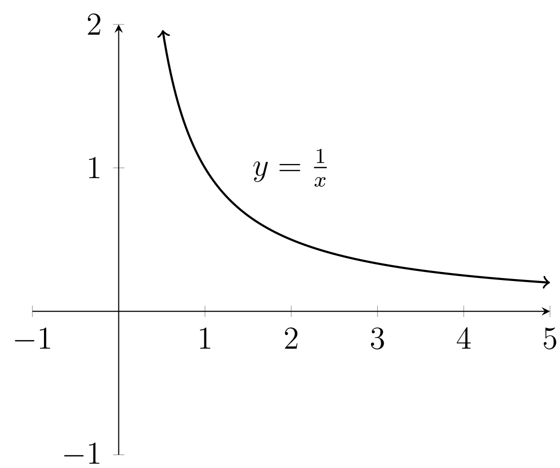


We can see that the slopes of these tangent lines are all different. In the case of $y = 2^x$, the slope of the tangent line at $(0,1)$ is about 0.7, while for the graph of $y = 3^x$, the slope of the tangent line at $(0,1)$ is about 1.1.

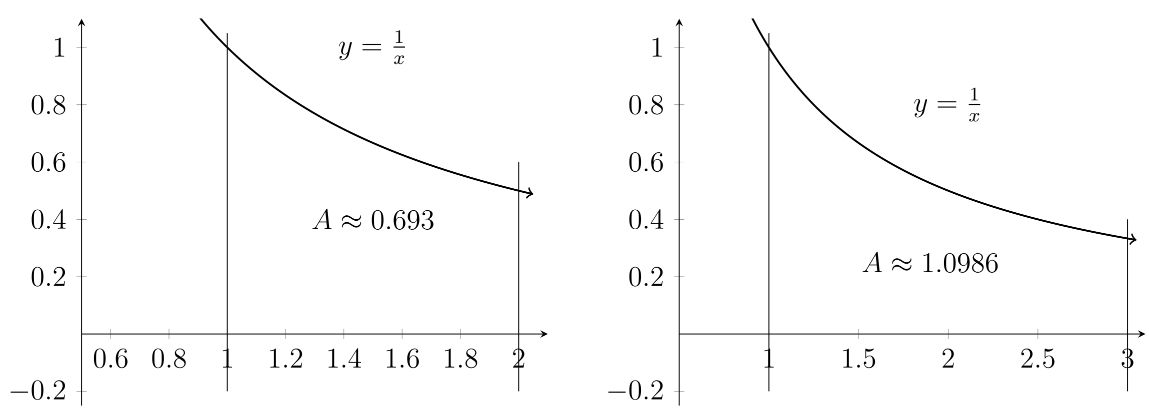
As mathematicians examined these graphs during the 17th and 18th centuries, they began to question what the value of the base "b" should be in the equation $y = b^x$ so that the slope of the tangent line at the point $(0, 1)$ would be equal to exactly 1. The answer was $e \approx 2.71828$.

Another way to derive the value of e uses Integral Calculus. Integral Calculus is often concerned with finding the area under a curve. This process can then be generalized and used to make many other types of calculations that are similar to finding area.

Consider the graph of the curve $y = \dfrac{1}{x}$:



We can delineate borders on the x values and determine the area of the resulting region using the techniques of calculus:

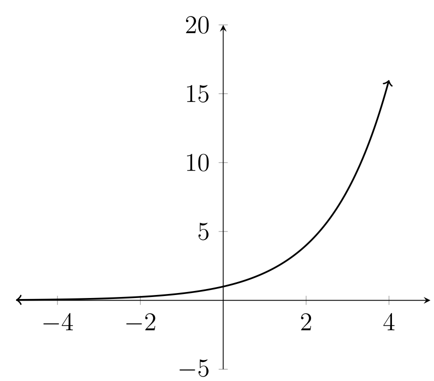


These values for the area under the curve are actually the same values as those for the slope of the tangent line in the previous graphs. If you ask the question, "Where should you draw the second vertical line so that the area under the curve is equal to exactly 1?" then just like the slope question, the answer is $e \approx 2.71828$.

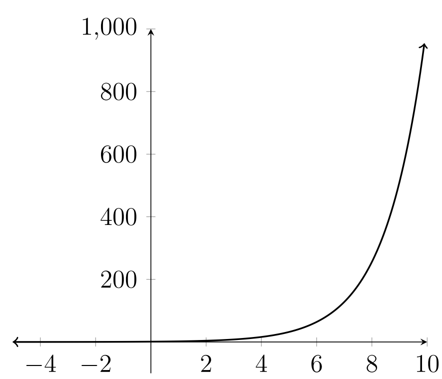
This is how the value of e was determined and why it is used to represent these exponential relationships.

Logistic Relationships

Let's consider the graph of $y = 2^x$:



If we extend the x-axis out further past $x = 4$, we would see that the y values for this relationship will grow *very* quickly, as they continue doubling.



Some phenomena in the natural world exhibit behavior similar to the growth of this function. However, in the natural world, few, if any, things can grow unconstrained. Most growth of any kind is limited by the resources that fuel the growth. Populations often grow exponentially for a period of time, however, populations are dependent on natural resources to continue growing. As a result, the simple exponential function is only useful for modeling real-world behavior if the x-values are limited.

It was this problem with the simple exponential function that led French mathematician Pierre Verhulst to slightly adjust the differential equation that gives rise to the exponential function to make it more realistic.

The original differential equation said:

$$y' = k * y$$

This says that the rate of growth of y is *always* directly proportional to the value of y. In other words the larger a population gets, the faster it will grow - forever. Verhulst changed this to say:

$$y' = k * y(1 - \frac{y}{N})$$

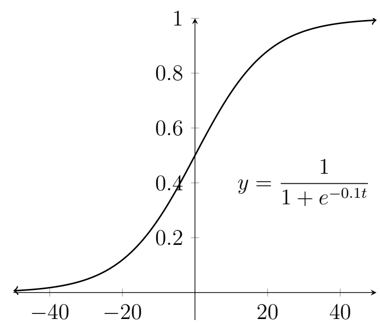
This is the defining relationship for the Logistic function. Notice that when values of y are small, this is essentially the same as the simple exponential. If y is small, then the $(1 - \frac{y}{N})$ term will be very close to 1 and will produce behavior very much like the simple exponential.

The N in the formula is a theoretical "maximum population." As the value of y approaches this maximum value, $\frac{y}{N}$ will approach 1 and $(1 - \frac{y}{N})$ will get smaller and smaller. As it gets smaller, the factor of $(1 - \frac{y}{N})$ will slow down the growth of the function to model the pressure that is put on the resources that are driving the growth.

The solution of the Logistic equation is quite complicated and results in a standard form of:

$$y = \frac{N}{1 + be^{-kt}}$$

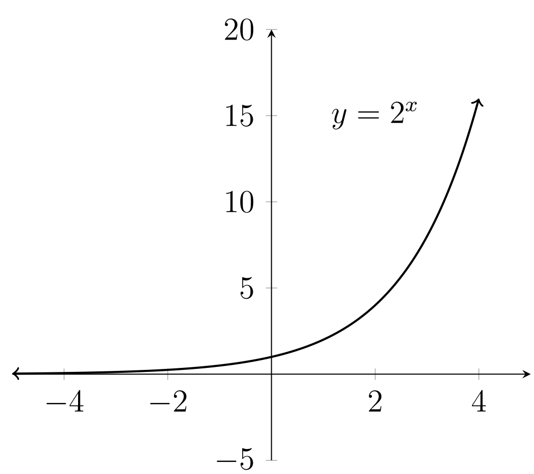
The graph for a sample logistic relationship is shown below:



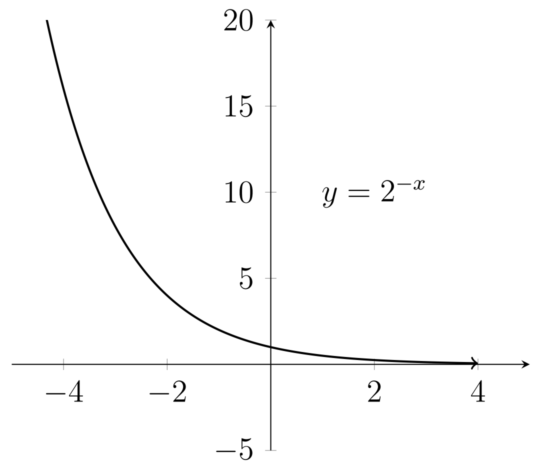
The "lazy-s" shape is characteristic of the logistic function. In the early stages, the relationship shows growth very similar to the simple exponential function, but, as the function grows larger, the growth decreases and the function values stabilize. The maximum y value of N is always the horizontal asymptote for the logistic function.

Negative Exponential Relationships

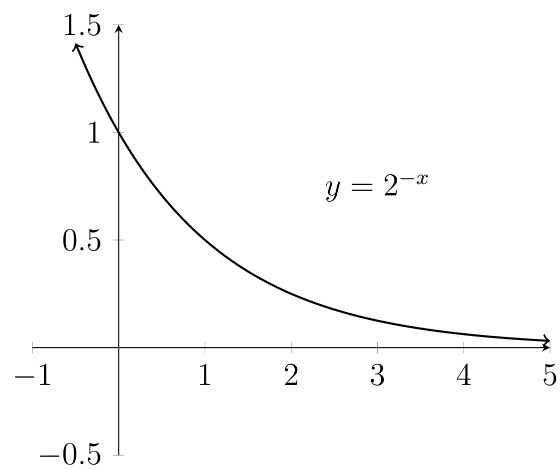
The Logistic function is very useful for modeling phenomena from the natural world. Although the simple exponential function is somewhat limited in modeling natural phenomena, the *negative* exponential is quite useful. Looking back to the graph of $y = 2^x$:



If we turn the graph around by changing x to $-x$ in the formula, then we will be working with the decaying tail of the graph:



Let's zoom in on the portion of the graph in the Quadrant I:



The behavior shown in the graph is quite useful for modeling radioactive decay, processes of heating and cooling, and assimilation of medication in the bloodstream. Let's look at an example.

Example

If a patient is injected with 150 micrograms of medication, the amount of medication still in the bloodstream after t hours is given by the function:

$$A(t) = 150e^{-0.12t}$$

a) Find the amount of medication in the bloodstream after 3 hours.

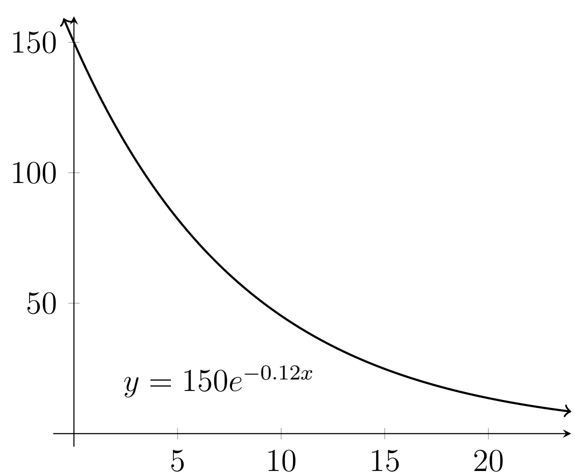
Round your answer to the nearest 100th.

b) How long will it take for the amount of medication to reach 60 micrograms?

Round your answer to the nearest 10th of an hour.

In this section we will focus on solving these problems using the graphing calculator. In later sections, we will cover processes that can be used to solve these problems algebraically. It is helpful to understand both methods of solution.

First, let's graph the function given in the problem:

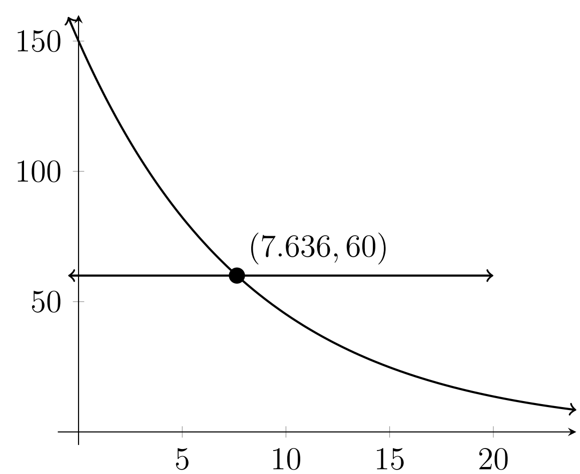


We can directly calculate the answer for part (a) by plugging the value of 3 for t or we can use the table in the graphing calculator to find this value:

x	y
0	150
1	133.04
2	117.99
3	104.65

We can see that after 3 hours there are approximately 104.65 micrograms of medication in the patient's bloodstream.

To answer part (b) graphically, we'll graph the original function along with the horizontal line $y = 60$. When we find the intersection of these two graphs, we'll know how long it takes for there to be 60 micrograms of medication in the patient's bloodstream:



Here, we can see that it would take about 7.6 hours (or about 7 hours 36 minutes), for there to be 60 micrograms of medication in the patient's bloodstream.

Example

The deer population on a nature preserve can be modeled using the equation:

$$y = \frac{8000}{1 + 9e^{-0.2t}}$$

y indicates the number of deer living in the nature preserve and t represents the number of years that have passed since the initial population of deer were established there.

a) How many deer were in the initial population?

b) What is the deer population after 10 years?

Round your answer to the nearest whole number.

c) How long does it take for the population to reach 5,000?

Round your answer to the nearest 10th of a year.

Parts (a) and (b) can both be answered from a table of values for the function. Before we look at the table, let's consider the question in Part (a). The problem is asking what the initial population of deer was. The means that we're looking to find out what y is when $t = 0$.

Let's see what happens when we plug zero in for t in the formula:

$$y = \frac{8000}{1 + 9e^{-0.2*0}} = \frac{8000}{1 + 9e^{0}}$$

$$= \frac{8000}{1 + 9 * 1} = \frac{8000}{10}$$

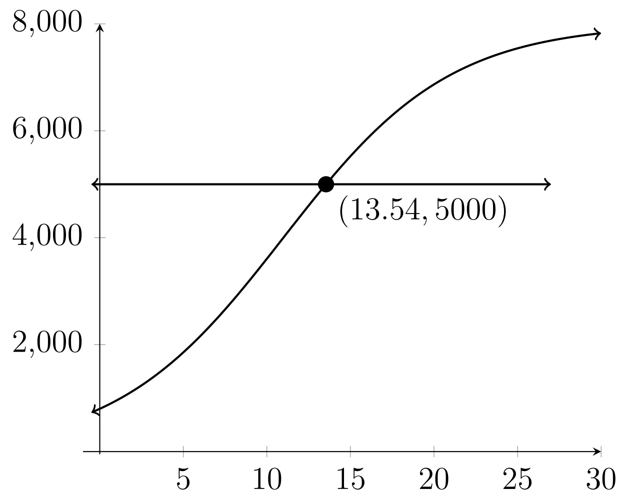
$$= 800$$

Let's look at the table of values:

t	y
0	800
5	1855.8
10	3606.8

To the nearest whole number, the deer population after 10 years is 3,607.

In Part (c), we'll need to graph a horizontal line at $y = 5,000$ and find the intersection.



To the nearest 10th, it takes about 13.5 years for the deer population to reach 5,000.

Exercises 3.1

1) **Medication in the bloodstream**

The function $A(t) = 200e^{-0.014t}$ gives the amount of medication, in milligrams, in a patient's bloodstream t minutes after the medication has been injected.

a) Find the amount of medication in the bloodstream after 45 minutes.

Round your answer to the nearest milligram.

b) How long will it take for the amount of medication to reach 50 milligrams?

Round your answer to the nearest minute.

2) **Medication in the bloodstream**

The function $D(t) = 50e^{-0.2t}$ gives the amount of medication, in milligrams, in a patient's bloodstream t hours after the medication has been injected.

a) Find the amount of medication in the bloodstream after 3 hours.

Round your answer to the nearest milligram.

b) How long will it take for the amount of medication to reach 10 milligrams?

Round your answer to the nearest minute.

3) **Fish Population**

The number of bass in a lake can be modeled using the given equation:

$$P(t) = \frac{3600}{1 + 7e^{-0.05t}}$$

where t is the number of months that have passed since the lake was stocked with bass. In each question below, round your answer to the nearest whole number.

a) How many bass were in the lake immediately after it was stocked?

b) How many bass were in the lake 1 year after it was stocked?

4) **Bird Population**

The population of a certain species of bird is limited by the type of habitat required for nesting. The population can be modeled using the following equation:

$$P(t) = \frac{5600}{0.5 + 27.5e^{-0.044t}}$$

where t is the number of years. In each question below, round your answer to the nearest whole number.

a) Find the initial bird population.

b) What is the population 100 years later?

5) **A Temperature Model**

A cup of coffee is heated to $180°F$ and placed in a room that maintains a temperature of $65°F$. The temperature of the coffee after t minutes have passed is given by:

$$F(t) = 65 + 115e^{-0.042t}$$

a) Find the temperature of the coffee 10 minutes after it is placed in the room.

Round your answer to the nearest degree.

b) When will the temperature of the coffee be $100°F$?

Round your answer to the nearest tenth of a minute.

6) **A Temperature Model**

Soup at a temperature of $170°F$ is poured into a bowl in a room that maintains a constant temperature. The temperature of the soup decreases according to the model given by:

$$F(t) = 75 + 95e^{-0.12t}$$

where t is the number of minutes that have passed since the soup was poured.

a) What is the temperature of the soup after 2 minutes?

Round your answer to the nearest 10th of a degree.

b) A certain customer prefers that the soup be cooled to $110°F$.

How long will this take?

Round your answer to the nearest 10th of a minute.

7) **Radioactive Decay**

A radioactive substance decays in such a way that the amount of mass remaining after t days is given by the equation:

$$m(t) = 13e^{-0.015t}$$

where the amount is measured in kilograms.

a) Find the mass at time $t = 0$.

b) How much of the mass remains after 45 days?

c) How long does it take for there to be 5 kg. left?

8) **Radioactive Decay**

Radioactive iodine is used by doctors as a tracer in diagnosing certain thyroid gland disorders. This type of iodine decays in such a way that the mass remaining after t days is given by:

$$m(t) = 6e^{-0.087t}$$

where the amount is measured in grams.

a) Find the mass at time $t = 0$.

b) How much remains after 20 days?

c) How long does it take for there to be 2 grams left?

Round your answer to the nearest tenth of a day.

9) **Fish population**

The function:

$$P(t) = \frac{12}{1 + e^{-t}}$$

gives the size of a fish population in thousands at time t, measured in years.

a) Find the initial population of fish at time $t = 0$.

Find the population of fish after 2 years, time $t = 2$.

b) How long will it take for the population to be 10,000?

What appears to be the maximum population for this particular model?

10) **Fish population**

The function:

$$P(t) = \frac{70}{1 + 2e^{-t}}$$

gives the size of a fish population in thousands at time t, measured in years.

a) Find the initial population of fish at time $t = 0$.

Find the population of fish after 1.5 years, $t = 1.5$.

b) How long will it take for the population to be 50,000?

What appears to be the maximum population for this particular model?

11) **Fish population**

The function:

$$P(t) = \frac{20}{1 - 0.4e^{-0.35t}}$$

gives the size of a fish population in thousands at time t, measured in years.

a) Find the initial population of fish at time $t = 0$.

Find the population of fish after 5 years, time $t = 5$.

b) How long will it take for the population to be 25,000?

What happens to this population over time?

12) **Fish population**

The function:

$$P(t) = \frac{10}{1 - 0.73e^{-0.08t}}$$

gives the size of a fish population in thousands at time t, measured in years.

a) Find the initial population of fish at time $t = 0$.

Find the population of fish after 5 years, time $t = 5$.

b) How long will it take for the population to be 25,000?

What happens to this population over time?

13) **Continuous Mixing**

A 100 gallon tank of pure water has salt water with 0.5 lb per gallon added to it at a rate of 2 gallons per minute. The brine solution is mixed thoroughly and drained at rate of 2 gallons per minute.

The equation for how many pounds of salt are in the tank at time t is given by:

$$y = 50 - 50e^{-0.02t}$$

where y is measured in pounds and t is measured in minutes.

a) How many pounds of salt are in the tank after 20 minutes?

b) How long does it take for there to be 30 lbs of salt in the tank?

Round your answer to the nearest tenth of a minute.

14) **Continuous Mixing**

A 20 gallon tank of pure water has salt water with 0.4 lb per gallon of salt added to it at a rate of 3 gallons per minute. The brine solution is mixed thoroughly and drained at rate of 3 gallons per minute.

The equation for how many pounds of salt are in the tank at time t is given by:

$$y = 8 - 8e^{-0.15t}$$

where y is measured in pounds and t is measured in minutes.

a) How many pounds of salt are in the tank after 15 minutes?

b) How long does it take for there to be 6 lbs of salt in the tank?

Round your answer to the nearest tenth of a minute.

15) **Continuous Mixing**

A tank contains 200 gallons of a 2% solution of HCl. A 5% solution of HCl is added at 5 gallons per minute. The well mixed solution is being drained at 5 gallons per minute.

The amount of HCl in the tank at any given time t in minutes is:

$$y = 10 - 6e^{-0.025t}$$

a) How many gallons of HCl are in the tank at $t = 25$ minutes?

b) When does the concentration of HCl in the solution reach 4%?

16) **Continuous Mixing**

A tank contains 100 gallons of salt water which contains a total of 25 lbs of salt. Salt water containing 0.4 lbs per gallon is added to the tank at a rate of 5 gallons per minute and the well-mixed solution is drained at the same rate.

The amount of salt in pounds in the tank at any given time t in minutes is:

$$y = 40 - 15e^{-0.05t}$$

a) How many pounds of salt are in the tank at $t = 20$ minutes?

b) How long does it take for there to be 30 lbs of salt in the tank?

3.2 Logarithmic Notation

A Logarithm is an exponent. In the early 1600's, the Scottish mathematician John Napier devised a method of expressing numbers in terms of their powers of ten in order simplify calculation. Since the advent of digital calculators, the methods of calculation using logarithms have become obsolete, however the concept of logarithms continues to be used in many area of mathematics.

The fundamental idea of logarithmic notation is that it is simply a restatement of an exponential relationship. The definition of a logarithm says:

$$\log_b N = x \to b^x = N$$

The notation above would be read as "log to the base b of N equals x means that b to the x power equals N." In this section we will focus mainly on becoming familiar with this notation. In later sections, we will learn to use this process to solve equations.

Example

Express the given statement using exponential notation:

$\log_2 32 = 5$

If $\log_2 32 = 5$, then $2^5 = 32$.

Example

Express the given statement using exponential notation:

$\log_7 4 \approx 0.7124$

If $\log_7 4 \approx 0.7124$, then $7^{0.7124} \approx 4$

If the logarithm notation appears without a base, it is usually assumed that the base should be 10.

Example

Express the given statement using exponential notation:

$\log 100 = 2$

If $\log 100 = 2$, then $10^2 = 100$

The notation $\ln N = x$ is typically used to indicate a logrithm to the base e. This means that:

$$\ln N = x \rightarrow e^x = N$$

Example

Express the given statement using exponential notation:

$\ln 15 \approx 2.708$

If $\ln 15 \approx 2.708$, then $e^{2.708} \approx 15$

In some cases, we would want to change an exponential statement into a logarithmic statement.

Exmple

Express the given statement using logarithmic notation:

$12^4 = 20,736$

If $12^4 = 20,736$ then $\log_{12} 20,736 = 4$

Example

Express the given statement using logarithmic notation:

$10^{2.5} \approx 316.23$

If $10^{2.5} \approx 316.23$ then $\log 316.23 \approx 2.5$

Example

Express the given statement using logarithmic notation:

$e^6 \approx 403.4$

If $e^6 \approx 403.4$, then $\ln 403.4 \approx 6$

Exercises 3.2

Rewrite each of the following using exponential notation.

1) $t = \log_5 9$

2) $h = \log_7 10$

3) $\log_5 25 = 2$

4) $\log_6 6 = 1$

5) $\log 0.1 = -1$

6) $\log 0.01 = -2$

7) $\log 7 \approx 0.845$

8) $\log 3 \approx 0.4771$

9) $\log_2 35 \approx 5.13$

10) $\log_{12} 50 \approx 1.5743$

11) $\ln 0.25 \approx -1.3863$

12) $\ln 0.989 \approx -0.0111$

Rewrite each of the following using logarithmic notation.

13) $10^2 = 100$

14) $10^4 = 10,000$

15) $4^{-5} = \dfrac{1}{1024}$

16) $5^{-3} = \dfrac{1}{125}$

17) $16^{\frac{3}{4}} = 8$

18) $8^{\frac{1}{3}} = 2$

19) $10^{1.3} \approx 20$

20) $10^{0.301} = 2$

21) $e^3 \approx 20.0855$

22) $e^2 \approx 7.3891$

23) $e^{-4} \approx 0.0183$

24) $e^{-2} \approx 0.1353$

3.3 Solving Exponential Equations

Because of the fact that logarithms are exponents, the rules for working with logarithms are similar to those that govern exponential expressions. One very helpful rule of equality for working with logarithms is related to the exponential rule for raising a power to a power. We recall one of the rules of exponents as:

$$(b^x)^y = b^{x*y}$$

in other words

$$(5^2)^4 = (5^2)(5^2)(5^2)(5^2) = 5^{2*4} = 5^8$$

In logarithmic notation, this rule works out as:

$$\log_b M^p = p * \log_b M$$

The reason for this comes from the rule for exponents. Lets' say that $\log_b M = x$.

Then:

$$\log_b M = x$$

this means that

$$b^x = M$$

and

$$(b^x)^p = (M)^p$$

so

$$b^{p*x} = M^p$$

Now, we come back to the question of $\log_b M^p = ?$. This expression ($\log_b M^p$) is asking the question "What power do we raise b to in order to get an answer of M^p? The result on the previous page shows that:

$$b^{px} = M^p$$

This means that we must raise b to the px power to get an answer of M^p. Remember that $x = \log_b M$. This means that:

$$b^{px} = M^p$$

so

$$\log_b M^p = px = p * \log_b M$$

This statement of equality is useful if we are trying to solve equations in which the variable is an exponent.

Example

Solve for x.

$4^x = 53$

We start by taking a logarithm on both of the equation. Just as we can add to both sides of an equation, or multiply on both sides of an equation, or raise both sides of an equation to a power, we can also take the logarithm of both sides. So long as two quantities are equal, then their logarithms will also be equal.

$$4^x = 53$$

$$\log 4^x = \log 53$$

$$x \log 4 = \log 53$$

$$x = \frac{\log 53}{\log 4} \approx 2.864$$

Since the log base 10 and log base e are both programmed into most calculators, these are the most commonly used bases for logarithms.

Example

Solve for x.

$5^{2x+3} = 17$

We start this problem in the same fashion, but this time we will use a logarithm to the base e:

$$5^{2x+3} = 17$$

$$\ln 5^{2x+3} = \ln 17$$

$$(2x+3)\ln 5 = \ln 17$$

There are several possibilities for finishing the problem from this point. We will focus on two of them that are the most useful for solving more complex problems. First, we will distribute the $\ln 5$ into the parentheses and then get the x by itself.

$$(2x+3)\ln 5 = \ln 17$$

$$x * 2\ln 5 + 3\ln 5 = \ln 17$$

$$-3\ln 5 = -3\ln 5$$

$$x * 2\ln 5 = \ln 17 - 3\ln 5$$

$$x = \frac{\ln 17 - 3\ln 5}{2\ln 5} \approx -0.620$$

And we can check the answer by plugging it back in:

$$5^{2*(-0.0620)+3} \approx 5^{1.760} \approx 16.9897 \approx 17$$

We can also approximate the logarithms in the problem and solve for an approximate answer:

$$(2x+3)\ln 5 = \ln 17$$

$$x * 2\ln 5 + 3\ln 5 = \ln 17$$

$$3.2189x + 4.8283 \approx 2.8332$$

$$-4.8283 \approx -4.8283$$

$$3.2189x \approx -1.9951$$

$$x \approx -0.620$$

If you use the method of approximating, it's important to make a good approximation. At least 4-5 decimal places are necessary for an accurate answer.

Let's look at an example that has variables on both sides of the equation:

Example

Solve for x.

$4^{3x} = 9^{2x-1}$

We'll use log base 10 in this problem.

$$4^{3x} = 9^{2x-1}$$

$$\log 4^{3x} = \log 9^{2x-1}$$

$$3x * \log 4 = (2x-1)\log 9$$

$$x * 3\log 4 = x * 2\log 9 - \log 9$$

If we collect like terms, we'll end up with:

$$x * 3\log 4 = x * 2\log 9 - \log 9$$

$$\log 9 = x * 2\log 9 - x * 3\log 4$$

At this point, if we want to get the x by itself, we need to factor out the x on the right-hand side:

$$\log 9 = x * 2\log 9 - x * 3\log 4$$

$$\log 9 = x(2\log 9 - 3\log 4)$$

Then divide on both sides by the coefficient in parentheses:

$$\frac{\log 9}{2\log 9 - 3\log 4} = \frac{x\cancel{(2\log 9 - 3\log 4)}}{\cancel{2\log 9 - 3\log 4}}$$

$$\frac{\log 9}{2\log 9 - 3\log 4} = x$$

$$9.327 \approx x$$

Again, we can check our answer by plugging it back into the equation:

$$4^{3*9.327} \approx 4^{27.981} \approx 7.0184 * 10^{16}$$

$$9^{2*9.327-1} \approx 9^{17.654} \approx 7.0177 * 10^{16}$$

We could also have solved this equation by approximating the logarithms in the beginning.

$$4^{3x} = 9^{2x-1}$$

$$\log 4^{3x} = \log 9^{2x-1}$$

$$3x * \log 4 = (2x-1)\log 9$$

$$3x(0.60206) \approx (2x-1)0.95424$$

$$1.80618x \approx 1.9085x - 0.95424$$

$$0.95424 \approx 0.10232x$$

$$9.326 \approx x$$

This answer is less accurate than the other approximation (9.326036 vs. 9.327424). The accuracy of an answer depends upon the original approximations for the logarithms.

Exercises 3.3

Solve for the indicated variable.

1) $2^x = 5$

2) $2^x = 9$

3) $3^x = 7$

4) $3^x = 20$

5) $2^{x+1} = 6$

6) $7^{x+1} = 41$

7) $5^{x+1} = 36$

8) $8^{x-2} = 6$

9) $4^{2x+3} = 50$

10) $4^{x+2} = 5^x$

11) $5^{2x+1} = 9$

12) $6^{x+4} = 10^x$

13) $7^{y+1} = 3^y$

14) $2^{x+1} = 3^{x-2}$

15) $6^{y+2} = 5^y$

16) $7^{x-3} = 3^{x+1}$

17) $6^{2x+1} = 5^{x+2}$

18) $9^{1-x} = 12^{x+1}$

19) $5^{2x-1} = 3^{x-3}$

20) $3^{x-2} = 4^{2x+1}$

21) $8^{3x-2} = 9^{x+2}$

22) $2^{2x-3} = 5^{-x-1}$

23) $10^{3x+2} = 5^{x+3}$

24) $5^{3x} = 3^{x+4}$

25) $3^{x+4} = 2^{1-3x}$

26) $4^{2x+3} = 5^{x-2}$

27) $3^{2-3x} = 4^{2x+1}$

28) $2^{2x-3} = 5^{x-2}$

3.4 Solving Logarithmic Equations

In the previous section, we took exponential equations and used the properties of logarithms to restate them as logarithmic equations. In this section, we will take logarithmic equations and use properties of logarithms to restate them as exponential equations. In the previous section, we used the property of logarithms that said $\log_b M^p = p \log_b M$. In this section, we will make use of two additional properties of logarithms:

$$\log_b(M * N) = \log_b M + \log_b N \qquad \text{and} \qquad \log_b \tfrac{M}{N} = \log_b M - \log_b N$$

Just as our previous property of logarithms was simply a restatement of the rules of expoenents, these two properties of logarithms depend on the rules of exponents as well. Since we're interested in $\log_b M$ and $\log_b N$, let's restate these in terms of exponents:

$$\text{If } \log_b M = x \text{ then } b^x = M \text{ and if } \log_b N = y \text{ then } b^y = N$$

The properties of logarithms we're interested in justifying have to do with $M * N$ and $\frac{M}{N}$, so let's look at those expressions in terms of exponents:

$$M * N = b^x * b^y = b^{x+y}$$

and

$$\frac{M}{N} = \frac{b^x}{b^y} = b^{x-y}$$

If we're interested in $\log_b(M * N)$, then we're asking the question "What power do we raise b to in order to get $M * N$?" We can see above that raising b to the $x + y$ power gives us $M * N$. Since $x = \log_b M$ and $y = \log_b N$ then $x + y = \log_b M + \log_b N$, so:

$$\log_b(M * N) = x + y = \log_b M + \log_b N$$

Likewise, if we're interested in $\log_b \frac{M}{N}$, we're asking the question "What power do we raise b to in order to get $\frac{M}{N}$?" Since raising b to the $x - y$ power gives us $\frac{M}{N}$ and $x - y = \log_b M - \log_b N$, then:

$$\log_b \tfrac{M}{N} = x - y = \log_b M - \log_b N$$

Let's look at an example to see how we'll use this to solve equations:

Example

Solve for x.

$\log_2 x + \log_2(x-4) = 2$

The first thing we can do here is to combine the two logarithmic statements into one. Since $\log_b(M * N) = \log_b M + \log_b N$, then $\log_2 x + \log_2(x-4) = \log_2 \left[x(x-4)\right]$.

$$\log_2 x + \log_2(x-4) = 2$$

$$\log_2 \left[x(x-4)\right] = 2$$

Then we'll restate the resulting logarithmic relationship as an exponential relationship:

$$2^2 = x(x-4)$$

$$4 = x^2 - 4x$$

$$0 = x^2 - 4x - 4$$

$$4.828, -0.828 \approx x$$

Most textbooks reject answers that result in taking the logarithm of a negative number, such as would be the case for $x \approx -0.828$. However, the logarithms of negative numbers result in complex valued answers, rather than an undefined quantity. For that reason, in this text, we will include all answers.

If a problem involves a difference of logarithms, we can use the other property of logarithms introduced in this section.

Example

Solve for x.

$\log(5x-1)-\log(x-2)=2$

Again, our first step is to restate the difference of logarithms using the property $\log_b \frac{M}{N}=\log_b M-\log_b N$:

$$\log(5x-1)-\log(x-2)=2$$

$$\log\left[\frac{5x-1}{x-2}\right]=2$$

We're working with a logarithm in base 10 in this problem, so in our next step we'll say:

$$\log\left[\frac{5x-1}{x-2}\right]=2$$

$$\frac{5x-1}{x-2}=10^2$$

Then multiply on both sides by $x-2$:

$$10^2=\frac{5x-1}{x-2}$$

$$(x-2)*100=\frac{5x-1}{\cancel{(x-2)}}*\cancel{(x-2)}$$

And, solve for x

$$100x - 200 = 5x - 1$$

$$95x = 199$$

$$x = \frac{199}{95}$$

In some equations, all of the terms are stated using logartihms. These equations often come out in a form that says $\log_b x = \log_b y$. If this is the case, we can then conclude that $x = y$.

It seems reasonable that if the exponent we raise b to in order to get x is the same exponent that we raise b to in order to get y, then x and y are the same thing.

Assume:

$$\log_b x = \log_b y$$

Let's say that $\log_b x = a = \log_b y$

then

$b^a = x$ and $b^a = y$

if both x and y are equal to b^a, then $x = y$

Example

Solve for x.

$\log_5(4 - x) = \log_5(x + 8) + \log_5(2x + 13)$

First, let's use the properties of logarithms to restate the equation so that there is only one logarithm on each side.

$$\log_5(4-x) = \log_5(x+8) + \log_5(2x+13)$$

$$\log_5(4-x) = \log_5\big[(x+8)(2x+13)\big]$$

Then, we'll use the property of logarithms we just discussed:

If $\log_b x = \log_b y$

then

$$x = y$$

$$\log_5(4-x) = \log_5\big[(x+8)(2x+13)\big]$$

$$4-x = (x+8)(2x+13)$$

$$4-x = 2x^2+29x+104$$

$$0 = 2x^2+30x+100$$

$$0 = 2(x+5)(x+10)$$

$$-5, -10 = x$$

Exercises 3.4

Solve for the indicated variable in each equation.

1) $\log_3 5 + \log_3 x = 2$

2) $\log_4 x + \log_4 5 = 1$

3) $\log_2 x = 2 + \log_2 3$

4) $\log_5 x = 2 + \log_5 3$

5) $\log_3 x + \log_3(x - 8) = 2$

6) $\log_6 x + \log_6(x - 5) = 1$

7) $\log(3x + 2) = \log(x - 4) + 1$

8) $\log(x - 1) - \log x = -0.5$

9) $\log_2 a + \log_2(a + 2) = 3$

10) $\log_3 x + \log_3(x - 2) = 1$

11) $\log_2 y - \log_2(y - 2) = 3$

12) $\log_2 x - \log_2(x + 3) = 2$

13) $\log_3 x + \log_3(x + 4) = 2$

14) $\log_4 u + \log_4(u + 1) = 1$

15) $\log 5 + \log 5 = \log 6$

16) $\ln x + \ln 4 = \ln 2$

17) $\log_7 x - \log_7 12 = \log_7 2$

18) $\log 2 - \log x = \log 8$

19) $\log_3 x - \log_3(x-2) = \log_3 4$

20) $\log_6 2 - \log_6(x-2) = \log_6 9$

21) $\log_4 x - \log_4(x-4) = \log_4(x-6)$

22) $\log_9(2x+7) - \log_9(x-1) = \log_9(x-7)$

23) $2\log_2 x = \log_2(2x-1)$

24) $2\log_4 y = \log_4(y+2)$

25) $2\log(x-3) - 3\log 2 = 1$

26) $2\log_5 7 - \log_5(x+1) = \log_5(2x-5)$

3.5 Applications of the Negative Exponential Function

At the beginning of Chapter 3, we worked with application problems and solved them using the graphing calculator. In this section, we will revisit some of these application problems and use the solution methods discussed in the previous sections to solve these problems algebraically.

Radioactive Decay

The decay of a radioactive element into its non-radioactive form occurs following a time line dictated by the "half-life" of the element. The half-life is the amount of time that it takes for half of the existing radioactive material to decay to its non-radioactive form.

Consider the equation:

$$A(t) = A_0 e^{-kt}$$

where $A(t)$ is the amount of material left at time t, A_0 is the amount present at $t = 0$, and k is a constant that can be determined based on the half-life of the material.

If we know that after one half-life, there will 50% of the radioactive material remaining, then we can say that:

$$0.5A_0 = A_0 e^{-kt_h}$$

where t_h is the half-life. To solve this equation for k, we would first divide on both sides by A_0:

$$\frac{0.5A_0}{A_0} = \frac{A_0 e^{-kt_h}}{A_0}$$

$$0.5 = e^{-kt_h}$$

Then take the natural logarithm of both sides and bring the exponent down in front of the expression as a coefficient:

$$0.5 = e^{-kt_h}$$

$$\ln(0.5) = \ln(e^{-kt_h})$$

$$\ln(0.5) = -kt_h * \ln(e)$$

$$\ln(0.5) = -kt_h * 1$$

$$-\frac{\ln(0.5)}{t_h} = k$$

The value of k can then be used in the equation $A(t) = A_0e^{-kt}$ to determine the amount of material left after any time t.

Example

The isotope Gold-198 (^{198}Au) is a type of gold sometimes used in medical applications and has a half-life of 2.7 days. How much of a 65 gram sample of ^{198}Au will be left after 6 days? How long would it take for there to be 10 grams left?

If we know the half-life, we can calculate the value of the constant k.

$$k = -\frac{\ln(0.5)}{t_h}$$

$$k = -\frac{\ln(0.5)}{2.7}$$

$$k \approx 0.2567$$

Now that we know the value of k, we can directly calculate the amount of ^{198}Au left after 6 days:

$$\begin{aligned} A(t) &= A_0 e^{-kt} \\ &= 65 e^{-0.2567 * 6} \\ &\approx 13.932 \end{aligned}$$

So, approximately 13.932 grams of ^{198}Au would be left after 6 days.

In order to calculate how long it takes for 10 grams of ^{198}Au to be left, we'll need to solve for t in the equation $A(t) = A_0 e^{-kt}$ with $A(t) = 10$:

$$\begin{aligned} A(t) &= A_0 e^{-kt} \\ 10 &= 65 e^{-0.2567t} \end{aligned}$$

First, we'll divide on both sides by 65:

$$\frac{10}{65} = \frac{\cancel{65} e^{-0.2567t}}{\cancel{65}}$$

$$\frac{10}{65} = e^{-0.2567t}$$

Then, take the natural logarithm on both sides:

$$\ln\left(\frac{10}{65}\right) = \ln\left(e^{-0.2567t}\right)$$

We'll calculate an approximate value for $\ln\left(\frac{10}{65}\right)$ and restate the right hand side of the equation using the properties of logarithms:

$$-1.872 \approx -0.2567t * \ln e$$

$$-1.872 \approx -0.2567t * 1$$

$$-1.872 \approx -0.2567t$$

$$7.3 \approx t$$

So, it would take about 7.3 days for the 65 grams of ^{198}Au to decay to 10 grams.

Newton's Law of Cooling

Newton's Law of Cooling states that the temperature of an object can be determined using the equation:

$$T = T_a + Ce^{-kt}$$

where T_a is the ambient temperature of the surrounding environment. The values of the constants C and k can often be calculated from given information.

Example

A bottle of soda at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F.

After half an hour, the soda has cooled to 61°F. What is the temperature of the soda after another half hour?

First, since the soda is placed in the refrigerator where the ambient temperature is 44°F, then $T_a = 44$. We also know that at $t = 0$, $T = 72$, which is the temperature of the soda can when it is first put in the refrigerator. This will allow us to calculate the constant C.

$$T = T_a + Ce^{-kt}$$

$$72 = 44 + Ce^{-k*0}$$

$$72 = 44 + Ce^{0} = 44 + C * 1$$

$$72 = 44 + C$$

$$28 = C$$

Now we know that $T_a = 44$ and $C = 28$. We can use the other piece of information from the problem to calculate the value of k. The problem states that after 30 minutes the soda has cooled off to 61°F. That means that when $t = 30$ (or $t = 0.5$ depending on which units you choose) the temperature T will be 61°F. We can then set up the equation to reflect this and calculate the vlaue of k:

$$T = 44 + 28e^{-kt}$$

$$61 = 44 + 28e^{-k*30}$$

First, we'll subtract 44 on both sides and divide by 28:

$$61 = 44 + 28e^{-k*30}$$

$$17 = 28e^{-30k}$$

$$\frac{17}{28} = e^{-30k}$$

Now, we'll take the natural logarithm on both sides to bring the $-30k$ down from the exponent:

$$\ln\left(\frac{17}{28}\right) = \ln\left(e^{-30k}\right)$$

$$\ln\left(\frac{17}{28}\right) = -30k * \ln e$$

$$-0.499 \approx -30k$$

$$0.01663 \approx k$$

Now, we have the full formula for calculating temperature in this scenario:

$$T = 44 + 28e^{-0.01663t}$$

To find out what happens after 60 minutes, we can simply plug in 60 for t:

$$T = 44 + 28e^{-0.01663t}$$

$$T = 44 + 28e^{-0.01663*60}$$

$$T = 44 + 28e^{-0.9978}$$

$$T \approx 44 + 28 * 0.3687$$

$$T \approx 44 + 10.3236 \approx 54.3°F$$

Exercises 3.5

1) If the half-life of radioactive cesium-137 is 30 years, find the value of k in the equation $A(t) = A_0e^{-kt}$.

a) Given a 10 gram sample of cesium-137, how much will remain after 80 years?

b) How long will it take for only 2 grams of cesium-137 to remain?

2) The half-life for radioactive thorium-234 is about 25 days. Use this to find the value of k in the equation $A(t) = A_0e^{-kt}$.

a) How much of a 40 gram sample will remain after 60 days?

b) After how long will only 10 grams of thorium-234 remain?

3) Given a sample of strontium-90, it is known that after 18 years there are 32mg remaining and after 65 years there are 10mg remaining. Use this information to find out how much strontium-90 was in the sample to begin with, and also determine the half-life of strontium-90.

4) A 12 mg sample of radioactive polonium decays to 7.26 mg in 100 days.

a) What is the half-life of polonium?

b) How much of the 12 mg sample remains after 180 days?

5) A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling so that its temperature at time t is given by:

$$y = 65 + 145e^{-0.05t}$$

where t is measured in minutes and y is measured in degrees Fahrenheit.

a) What is the initial temperature of the soup?

b) What is the temperature after 10 minutes?

c) After how long will the temperature be $100°$?

6) Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is $98.6°$F. Immediately following death, the body begins to cool. This process uses Newton's Law of Cooling:

$$y = T_a + Ce^{-kt}$$

If the ambient temperature is $60°$, and the body has cooled to $72°$F after 6 hours, use this information to determine the value of k in the equation.

7) The police discover the body of a murder victim. Critical to solving the crime is determining when the murder was committed. The coroner arrives at the murder scene at 12 Noon. She immediately takes the temperature of the body and finds it to be $94.6°$F. She then takes the temperature 1.5 hours later and finds it to be $93.4°$F. If the temperature of the room is $70°$F, when was the murder committed?

8) A roasted turkey is taken from the oven when its temperature has reached $185°$F and is placed on a table in a room where the temperature is $75°$F.

a) If the temperature of the turkey is $150°$F after 30 minutes, what is its temperature after 45 minutes?

b) When will the turkey cool off to $100°$F?

9) A kettle full of water is brought to a boil in a room with an ambient temperature of $20°$C. After 15 minutes, the temperature of the water has decreased from $100°$C to $75°$C. Find an equation using Newton's Law of Cooling to represent to temperature at time t. Find the temperature of the water after 25 minutes.

10) A cup of coffee with a temperature of $105°$F is placed in a freezer with a temperature of $0°$F. After 5 minutes, the temperature of the coffee is $70°$F. Find an equation using Newton's Law of Cooling to represent to temperature at time t. What will the temperature be in 10 minutes?

Answer Key

Section 1.1

1) $-x^2 - 16x - 13$

3) $3b^2$

5) $6m - 3$

7) $-2a^2 + 19a - 58$

9) $2y^2 + 24y - 22$

Section 1.2

1) $8a^2b^2(b + 3)$

3) $13t^2(t^6 + 2t^2 - 3)$

5) $9mn^3(5m^3n^2 + 4n^3 + 9m)$

7) $(a + 2)(a + 1)$

9) $(x - 9)(x + 3)$

11) $(m + 9)(m - 6)$

13) $(a + 3)(a - 3)$

15) $(k + 7)(k - 7)$

17) $6(x + 3)(x - 3)$

19) $2(10 + a)(10 - a)$

21) $2(7 + k)(7 - k)$

23) $5(y + 4)(y - 4)$

25) $2(2y + 7)(2y - 7)$

27) $k(6 + 7k)(6 - 7k)$

Section 1.2(cont.)

29) PRIME

31) $(3x-4)(3x-2)$

33) $(2x+3)(x+2)$

35) $2(5y+3)(2y+1)$

37) $3(2a-3)(4a-1)$

39) $(10-y)(3+y)$

41) $(12+x)(2-x)$

43) $(14+x)(6-x)$

45) $3(2y^2+8y+5)$

47) $4a(5x+1)(x-2)$

Section 1.3

1) $x \approx 0.275, -7.275$

3) $x \approx 1.587, -0.420$

5) $x \approx 0.787, -1.905$

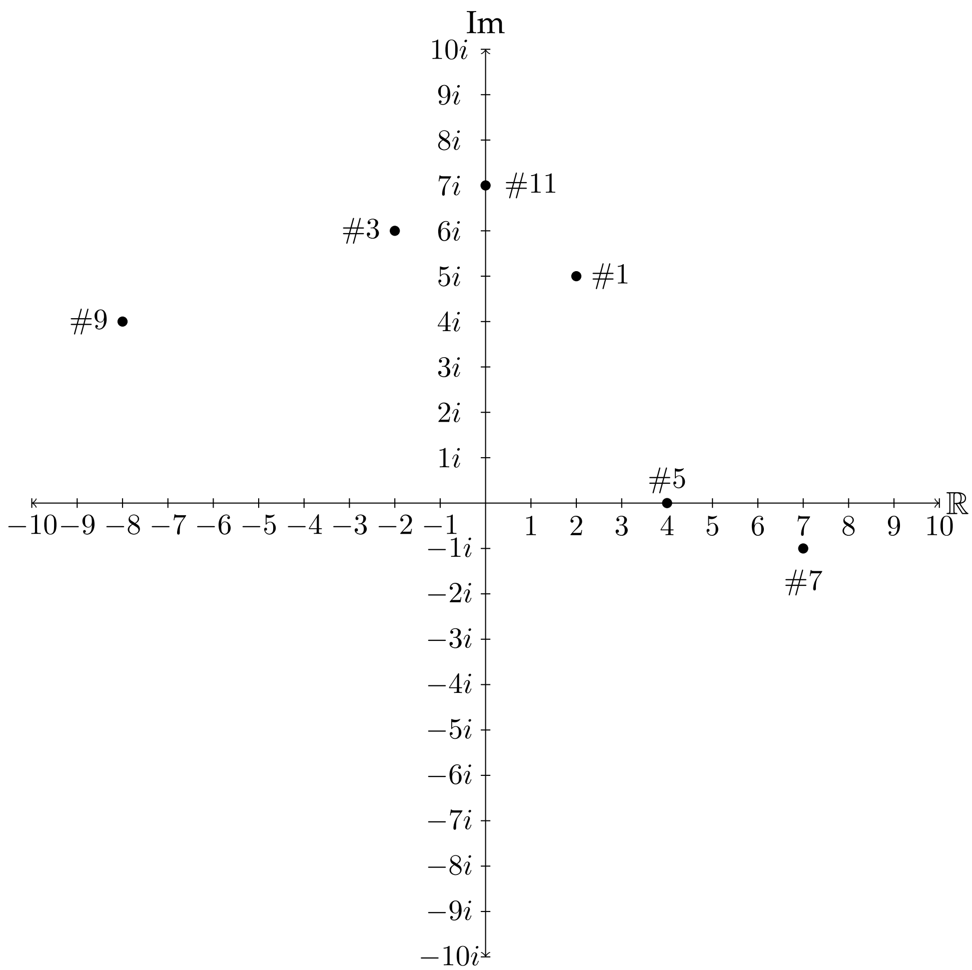
7) $x \approx -0.548, 2.002$

9) $x \approx -0.164, -1.172$

11) $x \approx -0.575, -2.175$

13) $x = 10, 5$

Section 1.4



13)	$6i$	15)	$10i$	17)	$2i$	19)	$1.414i$
21)	$3.162i$	23)	$2.236i$				

Section 1.4(cont.)

25)	$11 + 10i$	27)	$8 + 10i$	29)	$2 - i$	31)	-42
33)	10	35)	$1 + 5i$	37)	$27 - 28i$	39)	$11 + i$
41)	$12 - 16i$	43)	10	45)	97		
47)	$-i$	49)	i	51)	i	53)	1
55)	-1	57)	1	59)	$-i$	61)	-1

Section 1.5

1)	$x \approx 1.758, -0.758$	3)	$x \approx 0.344, -1.744$	5)	$x \approx 0.1\overline{6} \pm 0.373i$
7)	$x \approx -0.5 \pm 1.204i$	9)	$x \approx 0.286 \pm 0.881i$	11)	$x \approx 4 \pm 4.243i$

Section 1.6

1)	$\frac{3}{x-3}$	3)	$-\frac{x}{3}$	5)	$\frac{4x}{2x-1}$
7)	$-\frac{y+5}{2(y+1)}$	9)	$\frac{x+5}{x-1}$	11)	$-(x+7)$
13)	$\frac{3}{5x(x-2)}$	15)	$\frac{2a+3}{a+1}$	17)	$\frac{1}{x-y}$

Section 1.6 (cont.)

19) $\dfrac{4x}{(x-2)(x-3)}$

21) $3x$

23) $\dfrac{(x+1)(x-2)}{(x-1)(x-2)}$

25) $\dfrac{(x-4)(x+3)}{(x-1)(2x+1)}$

Section 1.7

1) $\dfrac{1}{x(x-1)}$

3) $\dfrac{6x-6}{x^2-9}$

5) $\dfrac{-k^2+5k+23}{(k+2)(k+5)}$

7) $\dfrac{-y^2-3y}{y^2-25}$

9) $\dfrac{x^2-2x+1}{(x+1)(x-3)}$

11) $\dfrac{b-1}{2b+2}$

13) $\dfrac{1}{a+1}$

15) $\dfrac{-1}{2x-4}$

17) $\dfrac{x}{x-2}$

19) $\dfrac{a}{a-3}$

21) $\dfrac{5x+3}{x+1}$

23) $\dfrac{5}{x-1}$

Section 1.8

1) $\dfrac{2}{2x+y}$

3) $\dfrac{m^2}{n(m-n)}$

5) $\dfrac{x}{y}$

7) $x+1$

9) $\dfrac{x+2}{x-2}$

11) $\dfrac{-5}{x^2-x-5}$

Section 1.8 (cont.)

13) $-\dfrac{2b-a}{3b-a}$

15) $\dfrac{(n+3)(n-2)}{n-1}$

17) $\dfrac{-x}{1-x}$ OR $\dfrac{x}{x-1}$

Section 1.9

1) $x=-1,-5$

3) $y \approx 3.449, -1.449$

5) $y=7$

7) $x=6,-4$

9) $n=12,-2$

11) $x=-0.2,3$

13) $x \approx -2.256, -5.911$

15) $x=0,16$

17) $x \approx 0.884, -1.884$

19) $y \approx 5.919, -2.253$

21) $a=0,\frac{2}{3}$

23) $x=-0.625$

25) $y=\frac{3}{7}$

27) $x \approx 13.659, -0.659$

29) $x=1$

Section 2.1

1) $-7<x\leq -4$ OR $0\leq x\leq 5$

3) $x\leq -1$ OR $x\geq 2$

5) $-8\leq x<-6$ OR $x\geq 3$

7) $x\leq -10$ OR $-4<x<1$ OR $x>6$

9) $-12\leq x<-7$ OR $x>5$

11) $x=0$ OR $x>4$

13) $x<-5$ OR $-3<x<-1$ OR $x\geq 4$

Section 2.2

1) $x \approx -0.4142, 2.4142, \frac{1}{3}$

3) $x \approx -4.035, 2.378, 2.657$

5) $x \approx 2.176, 2.902, -0.902, -0.176$

7) $x \approx -1.984, 0.707, 1.286, 4.991$

9) $x \approx 6.161, -0.537$

11) $x \approx -0.889, 1, 0.645, 1.745$

Section 2.3

1) $y \geq 0$: $-0.618 \leq x \leq 1.618$ OR $x \geq 3$

$y < 0$: $x < -0.618$ OR $1.618 < x < 3$

3) $y \geq 0$: $-1.5 \leq x \leq -0.236$ OR $x \geq 4.236$

$y < 0$: $x < -1.5$ OR $-0.236 < x < 4.236$

5) $y \geq 0$: $x \leq 1.5$ OR $x \geq \frac{5}{3}$

$y < 0$: $1.5 < x < \frac{5}{3}$

7) $y \geq 0$: $x \leq -2.656$ OR $-0.486 \leq x \leq 0.460$ OR $x \geq 1.682$

$y < 0$: $-2.656 < x < -0.486$ OR $0.460 < x < 1.682$

9) $x \leq -3$ OR $x = 1$

11) $x > 3$ OR $-4 < x < 1$

13) All real numbers

15) $x \leq -2.264$ OR $0.756 \leq x \leq 3.508$

17) $x < 0.5$ OR $x > 0.839$

Section 2.4

1) $-4 < x < 2$ OR $x > 6$

3) $-5 < x < -2$ OR $2 < x < 7$

5) $-\frac{2}{3} < x < 1.791$ OR $x < -2.791$

7) $-2.080 < x < -1.618$ OR $x > 0.618$

9) $-\frac{11}{3} < x < -2.162$ OR $x > 4.162$

11) $-2 < x < 1.791$ OR $x < -2.791$ OR $x > 3$

13) $-4.372 < x < -3.828$ OR $1.372 < x < 1.828$

15) $-5.193 < x < -2.646$ OR $0.193 < x < 2.646$

Section 2.5

1) $x^2 + 3x - 4 = 0$

3) $2x^2 - 5x + 3 = 0$

5) $3x^2 - 10x + 3 = 0$

7) $4x^2 - 12x - 7 = 0$

9) $3x^2 + 11x + 6 = 0$

11) $2x^2 - x + 15 = 0$

Section 2.6

1) $y^2 - 2y + 2$

3) $x - 5$

5) $x^3 - x + 3$

7) $(2z^2 - 2z + 7)$ $\quad R : 1$

9) $4x^2 + x + 1$

11) $(2y^3 - 3y^2 - 2y + 2)$ $\quad R : 3y + 2$

13) $(5x^2 + 15x + 17)$ $\quad R : -24x - 83$

Section 2.7

1) $x^2 - 3x - 10$

3) $x^2 + 5x - 1$

5) $x^3 - 4x^2 + x + 6$

7) $x^3 - 1 \quad R: 8$

9) $x^3 + 1$

11) $x^3 - 3x^2 + 4x - 2$

13) $x^2 + 4x + 5$

Section 2.8

1) $(x+1)(x-2)(x^2 - 2x + 5)$

$x = -1, 2, 1 \pm 2i$

3) $(x+1)(x-2)(2x^2 - 3x + 2)$

$x = -1, 2, 0.75 \pm 0.661i$

5) $(x-5)(x+2)(x^2 - 6x + 13)$

$x = 5, -2, 3 \pm 2i$

7) $(x-1)^3(x^2 - 6x + 10)$

$x = 1, 3 \pm i$

9) $(x-1)^3(x-3)$

$x = 1, 3$

11) $(x-1)^3(x^2 + 9)$

$x = 1, \pm 3i$

13) $(5x+1)(3x^2 - 2x + 3)$

$x = -\frac{1}{5}, \frac{1}{3} \pm 0.943i$

15) $(3x+2)(2x^2 + 3x + 2)$

$x = -\frac{2}{3}, -0.75 \pm 0.661i$

17) $(x+3)(2x-1)(2x^2 + 5x + 5)$

$x = -3, \frac{1}{2}, -1.25 \pm 0.968i$

19) $(2x+1)(x-3)(x^2 + x + 1)$

$x = 3, -\frac{1}{2}, -0.5 \pm 0.866i$

21) $(2x+1)(3x-2)(2x^2 + x + 1) \qquad x = \frac{2}{3}, -\frac{1}{2}, -0.25 \pm 0.661i$

Section 3.1

1) a) ≈ 107mg
b) ≈ 99 minutes

3) a) 450 bass
b) ≈ 744 bass

5) a) $\approx 141°F$
b) ≈ 28.3 minutes

7) a) 13 kg.
b) ≈ 6.62 kg.
c) ≈ 63.7 days

9) a) 6,000 10,570
b) 1.6 years 12,000

11) a) 33,333 21,494
b) ≈ 1.98 years
approaches 20,000

13) a) 16.5 lbs
b) 45.8 minutes

15) a) 6.8 gal.
43.94 minutes

Section 3.2

1) $5^t = 9$

3) $5^2 = 25$

5) $10^{-1} = 0.1$

7) $10^{0.845} \approx 7$

9) $2^{5.13} \approx 35$

11) $e^{-1.3863} \approx 0.25$

13) $\log 100 = 2$

15) $\log_4 \frac{1}{1024} = -5$

17) $\log_{16} 8 = \frac{3}{4}$

19) $\log 20 \approx 1.3$

21) $\ln 20.0855 \approx 3$

23) $\ln 0.0183 \approx -4$

Section 3.3

1) ≈ 2.322	3) ≈ 1.771	5) ≈ 1.585	7) ≈ 1.227
9) ≈ -0.089	11) ≈ 0.183	13) ≈ -2.297	15) ≈ -19.655
17) ≈ 0.723	19) ≈ -0.795	21) ≈ 2.117	23) ≈ 0.042
25) ≈ -1.165	27) ≈ 0.134		

Section 3.4

1) $x = 1.8$	3) $x = 12$	5) $x = 9, -1$
7) $x = 6$	9) $x = 2, -4$	11) $x = \frac{16}{7}$
13) $x \approx 1.606, -5.606$	15) $x = 1.2$	17) $x = 24$
19) $x = \frac{8}{3}$	21) $x = 3, 8$	23) $x = 1$

25) $x \approx 11.944, -5.944$

Section 3.5

1) a) ≈ 1.576g b) ≈ 69.67 years 3) 50mg

5) a) 210°F b) ≈ 152.9°F c) ≈ 28.4 minutes

7) 7:30AM 9) a) $T = 20 + 80e^{-0.025t}$ b) ≈ 62.85°C